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Some Embedding Theorems on the Nikolskii-Morrey Type Spaces

Ali Akbulut[1], Ahmet Eroglu[2] and Alik M. Najafov[34*]

[1]Department of Mathematics, Ahi Evran University, Kirsehir, Turkey
[2]Nigde University, Department of Mathematics, Nigde, Turkey
[3]Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
[4]Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
Email: nadjafov@rambler.ru

Abstract In the paper the Nikolskii-Morrey type spaces $H^l_{p,\varphi,\beta}(G)$ were introduced and studied. Some embedding theorems are obtained in $H^l_{p,\varphi,\beta}(G)$ with the help of Nikolskii type integral representation.

Keywords: Nikolskii-Morrey type spaces, integral representation, embedding theorems, generalized Holder condition

1 Introduction

In the paper, we introduce a Nikolskii-Morrey type space with parameters. By $H^l_{p,\varphi,\beta}(G)$ we denote the spaces of all functions $f \in L_1^{loc}(G)$ $(m_i > l_i - k_i > 0, i = 1, 2, ..., n)$ with the finite norm

$$\|f\|_{H^l_{p,\varphi,\beta}(G)} = \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n} \sup_{0<h<h_0} \frac{\left\| \Delta_i^{m_i}\left(\varphi_i\left(h\right), G_{\varphi(h)}\right) D_i^{k_i} f \right\|_{p,\varphi,\beta}}{\varphi_i\left(h\right)^{l_i-k_i}}, \tag{1.1}$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{x\in G, t>0} \left(\left|\varphi\left([t]_1\right)\right|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right), \tag{1.2}$$

$l \in (0,\infty)^n$, $m_i \in \mathbb{N}$, $k_i \in \mathbb{N}_0$, $p \in [1,\infty)$, $[t]_1 = \min\{1,t\}$, $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$, $\left|\varphi\left([t]_1\right)\right|^{-\beta} = \prod_{j=1}^{n} \left(\varphi_j\left([t]_1\right)\right)^{-\beta_j}$, $\beta_j \in [0,1]$, $j = 1, 2, ..., n$.

Denote by $\mathbb{A}$ the set of vector functions $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$ with Lebesgue measurable functions $\varphi_j(t) > 0$, $t > 0$, $\lim\limits_{t\to+0} \varphi_j(t) = 0$, $\lim\limits_{t\to+\infty} \varphi_j(t) = \infty$, $j = 1, 2, ..., n$.

For any $x \in R^n$,

$$G_{\varphi(t)}\left(x\right) = G \cap I_{\varphi(t)}\left(x\right) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), \quad j = 1, 2, ..., n \right\}.$$

Let for any $t > 0$, $\left|\varphi\left([t]_1\right)\right| \le C$, where C is positive constant. Then the embeddings $L_{p,\varphi,\beta}(G) \to L_p(G)$ and $H^l_{p,\varphi,\beta}(G) \to H^l_p(G)$ hold, i.e

$$\|f\|_{p,G} \le c\|f\|_{p,\varphi,\beta;G},$$
$$\|f\|_{H^l_p(G)} \le c\,\|f\|_{H^l_{p,\varphi,\beta}(G)}.$$

Note that the spaces $L_{p,\varphi,\beta}(G)$ and $H^l_{p,\varphi,\beta}(G)$ are Banach spaces. The completeness of these spaces automatically implies from completeness of L_p and H^l_p. The space $H^l_{p,\varphi,\beta}(G)$, when $\varphi_j(t) = t^{\chi_j}$, $\beta_j = \frac{a_j}{p}$ $(j = 1, ..., n)$ coincides with the space $H^l_{p,a,\chi}(G) \equiv H^l_{p,\lambda}$ introduced by J.Ross [13], in the case $\beta_j = 0\,(j = 1, ..., n)$ it coincides with the Nikolski space $H^l_p(G)$.The space $W^l_{p,\varphi,\beta}(G)$ was introduced and studied in [12]. The spaces of such type with different norms were introduced and studied in [2]-[11].

Note some properties of the spaces $L_{p,\varphi,\beta}(G)$.

1. The space $L_{p,\varphi,\beta}(G)$ is complete.

Proof. Let $\{f_i\}_{i=1}^{\infty}$ be the fundamental sequences in $L_{p,\varphi,\beta}(G)$, i.e. for any $i,j\to\infty$

$$\|f_i-f_j\|_{L_{p,\varphi,\beta}(G)}\to 0,$$

It means that $\forall\varepsilon>0$, $\exists n_0\in N$, $\forall i,j>n_0$

$$\|f_i-f_j\|_{L_{p,\varphi,\beta}(G)}<\varepsilon,$$

In other words $\forall\varepsilon>0$, $\exists n_0\in N$, $\forall i,j>n_0$

$$\sup_{x\in G,t>0}\left(|\varphi([t]_1)|^{-\beta}\|f_i-f_j\|_{p,G_{\varphi(t)}(x)}\right)<\varepsilon$$

and for any $x\in G$, $\forall t>0$

$$\|f_i-f_j\|_{p,G_{\varphi(t)}(x)}<\varepsilon$$

i.e. $\{f_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $L_p\left(G_{\varphi(t)}(x)\right)$. The space $L_p(G)$ is complete, therefore there is a function $f_0\in L_p(G), i\to\infty$, for $x\in G$, for any $t>0$, $\forall\varepsilon>0$

$$\|f_i-f_0\|_{L_p(G_{\varphi(t)}(x))}\to\varepsilon$$

then

$$|\varphi([t]_1)|^{-\beta}\|f_i-f_0\|_{L_p(G_{\varphi(t)}(x))}\to\varepsilon$$

i.e.

$$\|f_i-f_0\|_{p,\varphi,\beta;G}<\varepsilon,$$

$$\|f_0\|_{p,\varphi,\beta;G}=\|f_i-f_0\|_{L_{p,\varphi,\beta}(G)}+\|f_i\|_{L_{p,\varphi,\beta}(G)}<\varepsilon_1+M=\varepsilon_2$$

$\|f_0\|_{p,\varphi,\beta;G}<\varepsilon_2$,$f_0\in L_{p,\varphi,\beta}(G)$.

2. Let G be a bounded domain and $p\le q$; $\varphi(t)\le\psi(t)(t>0)$; $\exists c>0,\forall t\in(0,1),|\psi(t)|^{\beta_1}\le c|\varphi(t)|^{\beta}$, and then $L_{q,\psi,\beta_1}(G)\to L_{p,\varphi,\beta}(G)$ and there exists $C>0$ such that

$$\|f\|_{p,\varphi,\beta;G}\le C\|f\|_{q,\psi,\beta_1;G}.$$

Proof. . For any $t>0$, $x\in G$ we have

$$|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}$$

$$\le|\varphi([t]_1)|^{-\beta}(mesG_{\varphi(t)}(x))^{\frac{1}{p}-\frac{1}{q}}|\psi([t]_1)|^{\beta_1}\,|\psi([t]_1)|^{-\beta_1}\|f\|_{q,G_{\psi(t)}(x)}$$

and

$$\|f\|_{p,\varphi,\beta;G}\le C\|f\|_{q,\psi,\beta_1;G}.$$

Definition 1.1. The open set $G\subset R^n$ is said to be an open set with condition of flexible φ-horn if for some $\theta\in(0,1]^n$, $T\in(0,\infty)$ for any $x\in G$ there exists the vector-function

$$\rho\left(\varphi(t),x\right)=\left(\rho_1\left(\varphi_1(t),x\right),...,\rho_n\left(\varphi_n(t),x\right)\right),\quad 0\le t\le T$$

with the following properties:

1) For all $j=1,2,...,n$, $\rho_j\left(\varphi_j(t),x\right)$ are absolutely continuous on $[0,T]$, $|\rho_j\left(\varphi_j(t),x\right)|\le 1$ for almost all $t\in[0,T]$,

2) $\rho_j\left(0,x\right)=0,\ x+V(x,\theta)=x+\bigcup\limits_{0\le t\le T}\left[\rho\left(\varphi(t),x\right)+\varphi\left(t\right)\theta I\right]\subset G.$

In particular, for $\varphi(t)=t^{\lambda}$, $\left(t^{\lambda}=\left(t^{\lambda_1},t^{\lambda_2},...,t^{\lambda_n}\right)\right)$ and $\theta_j=\theta^{\lambda_j}$ $(j=1,\ldots,n)$ the set $x+V(x,\theta)$ is called the flexible λ-horn introduced in [1].

Assuming that $\varphi_j(t)$ $(j = 1,2,...,n)$ are also differentiable on $[0,T]$, we can show that for $f \in H_p^l(G)$ determined in n- dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x \in U \subset G)$

$$D^\nu f(x) = \overline{f}^{(\nu)}_{\varphi(t)}(x) + (-1)^{|\nu|} \sum_{i=1}^{n} \int_0^T \int_{R^n} \int_{-\infty}^{+\infty} K_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t,x))}{\varphi(t)} \right)$$
$$\times \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t),x) \right) \varDelta_i^{m_i}(\varphi_i(\delta)u)$$
$$\times f(x+y+ue_i) \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j-2} \frac{\varphi_i'(t)}{\varphi_i(t)} dtdudy, \tag{1.3}$$

$$\overline{f}^{(\nu)}_{\varphi(t)}(x) = \prod_{j=1}^{n} \varphi_j^{-2-\nu_j}(t) \int_{R^n}\int_{R^n} \varOmega^{(\nu)} \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)} \right)$$
$$\times \varOmega \left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)} \right) f(x+y+z)\, dydz. \tag{1.4}$$

Let $M_i(\cdot,y) \in C_0^\infty(R^n)$ be such that

$$S(M_i) \subset I_{\varphi(t)} = \left\{ y : |y_j| < \frac{1}{2}\varphi_j(t), \quad j = 1,2,...,n \right\}.$$

Assume $0 < T \leq 1$ is fixed and

$$V = \bigcup_{0<t\leq T} \left\{ y : \frac{y}{\varphi(t)} \in S(M_i) \right\}.$$

It is clear that $V \subset I_{\varphi(t)}$. Let $U + V \subset G$.

Lemma 1.2. *Let* $1 \leq p \leq q \leq r \leq \infty$; $0 < \eta,\ t < T \leq 1$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\nu_j \geq 0$ *are integers,* $j = 1,2,...,n$; $\varDelta_i^{m_i}(h) \in L_{p,\varphi,\beta}(G)$ *and let*

$$Q_T^i = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty,$$

$$A(x) = \prod_{j=1}^{n} \int_{R^n}\int_{R^n} f(x+y+z) \varOmega^{\nu} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)} \right)$$
$$\times \varOmega \left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)} \right) f(x+y+z)\, dydz, \tag{1.5}$$

$$H_\eta^i(x) = \int_0^\eta L_i(x,t) \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j-2} \frac{\varphi_i'(t)}{\varphi_i(t)} dt, \tag{1.6}$$

$$H_{\eta T}^i(x) = \int_\eta^T L_i(x,t) \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j-2} \frac{\varphi_i'(t)}{\varphi_i(t)} dt, \tag{1.7}$$

where

$$L_i(x,t) = \int_{R^n} \int_{-\infty}^{+\infty} M_i \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)} \right)$$
$$\times \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t),x)}{2\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi_i(t),x) \right) \varDelta_i^{m_i}(\varphi_i(\delta)u) f(x+y+ue_i)\, dudy \tag{1.8}$$

Then for any $\overline{x}\in U$ the following inequalities

$$\sup_{\overline{x}\in U}\left\|H^i_\eta\right\|_{qU_{\psi(\xi)}(\overline{x})}\le C_1\left\|\left(\varphi_i(t)\right)^{-l_i}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p,\varphi,\beta;G}$$

$$\times\left|Q^i_\eta\right|\prod_{j=1}^n\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p}{q}}, \tag{1.9}$$

$$\sup_{\overline{x}\in U}\left\|H^i_{\eta T}\right\|_{qU_{\psi(\xi)}(\overline{x})}\le C_2\left\|\left(\varphi_i(t)\right)^{-l_i}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p,\varphi,\beta;G}$$

$$\times\left|Q^i_{\eta T}\right|\prod_{j=1}^n\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p}{q}}, \tag{1.10}$$

$$\sup_{\overline{x}\in U}\|A\|_{qU_{\psi(\xi)}(\overline{x})}\le\|f\|_{p,\varphi,\beta;G}\prod_{j=1}^n\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_jp)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j=1}^n\left(\psi_j[\xi]_1\right)^{\beta_j\frac{p}{q}}, \tag{1.11}$$

is hold, where $U_{\psi(\xi)}(\overline{x})=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j(\xi),j=1,2,...,n\right\}$ and $\psi\in N$, C_1, C_2 are the constants independent of φ, ξ, η and T.

Proof. Applying sequentially the Minkowski generalized inequality for any $\overline{x}\in U$

$$\left\|H^i_\eta\right\|_{qU_{\psi(\xi)}(\overline{x})}\le\int_0^\eta\left\|L_i(\cdot,t)\right\|_{qU_{\psi(\xi)}(\overline{x})}\prod_{j=1}^n\left(\varphi_j(t)\right)^{\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt, \tag{1.12}$$

and from the Hölder inequality ($q\le r$) we have

$$\left\|L_i(\cdot,t)\right\|_{qU_{\psi(\xi)}(\overline{x})}\le\left\|L_i(\cdot,t)\right\|_{rU_{\psi(\xi)}(\overline{x})}\prod_{j=1}^n\left(\psi_j(\xi)\right)^{\frac{1}{q}-\frac{1}{r}}. \tag{1.13}$$

Now estimate the norm $\left\|L_i(\cdot,t)\right\|_{qU_{\psi(\xi)}(\overline{x})}$. Let X be a characteristic function of the set $S(M_i)=supp\,M_i$. Noting that $1\le p\le r\le\infty$, $s\le r$, represent the integrand function (1.8) in the form

$$\left|\int_{-\infty}^{+\infty}M_i\zeta_i\Delta_i^{m_i}f\,du\right|=\left(\left|\int_{-\infty}^{+\infty}\zeta_i\Delta_i^{m_i}f\,du\right|^p|M_i|^s\right)^{\frac{1}{r}}$$

$$\times\left(\left|\int_{-\infty}^{+\infty}\zeta_i\Delta_i^{m_i}f\,du\right|^pX\left(\frac{y}{\varphi(t)}\right)\right)^{\frac{1}{q}-\frac{1}{r}}\left(|M_i|^s\right)^{\frac{1}{s}-\frac{1}{r}}$$

and apply to $|L_i|$ the Hölder inequality $\left(\frac{1}{p}+\left(\frac{1}{p}-\frac{1}{r}\right)+\left(\frac{1}{s}-\frac{1}{r}\right)=1\right)$, we obtain

$$\left\|L_i(\cdot,t)\right\|_{r,U_{\psi(\xi)}(\overline{x})}\le\sup_{x\in U_{\psi(\xi)}(\overline{x})}\left(\int_{R^n}\left|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi(t),x\right)\right)\right.\right.$$

$$\left.\left.\times\Delta_i^{m_i}\left(\varphi_i(t)\right)f\left(x+y+ue_i\right)du\right|^p\chi\left(\frac{y}{\varphi(t)}\right)dy\right)^{\frac{1}{p}-\frac{1}{r}}$$

$$\times\sup_{y\in V}\left(\int_{U_{\psi(\xi)}(\overline{x})}\left|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi(t),x\right)\right)\right.\right.$$

$$\times \Delta_i^{m_i}(\varphi_i(t)u) f(x+y+ue_i)\,du|^p\,dx)^{\frac{1}{p}}$$
$$\times\left(\int_{R^n}\left|M_i\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\rho'(\varphi(t),x)\right)\right|^s dy\right)^{\frac{1}{s}}. \tag{1.14}$$

For any $x\in U$ we have

$$\int_{R^n}\left|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(t),x)\right)\right.$$
$$\times\Delta_i^{m_i}(\varphi_i(\delta)u) f(x+y+ue_i)\,du|^p\,\chi\left(\frac{y}{\varphi(t)}\right)dy$$
$$\le\int_{(U+V)_{\varphi(t)}(\overline{x})}\left|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(t),x)\right)\Delta_i^{m_i}(\varphi_i(\delta)u) f(y+ue_i)\,du\right|^p$$
$$\le(\varphi_i(t))^{pl_i}\left\|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(t),x)\right)\varphi_i(t)^{-l_i}\right.$$
$$\times\Delta_i^{m_i}\left(\varphi_i(\delta)u,G_{\varphi(t)}\right)fdu\|^p_{p,G_{\varphi(t)}(x)}$$
$$\le\varphi_i(t)^{p+pl_i}\left\|\varphi_i(t)^{-l_i}\Delta_i^{m_i}\left(\varphi_i(\delta)u,G_{\varphi(t)}\right)\right\|^p_{p,\varphi,\beta}\prod_{j=1}^n(\varphi_j(t))^{\beta_j p}. \tag{1.15}$$

For $y\in V$ and $U_\psi+V\subset G_\varphi$ $(\varphi([t]_1)\le\psi([t]_1))$

$$\int_{U_{\psi(\xi)}(\overline{x})}\left|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(t),x)\right)\Delta_i^{m_i}(\varphi_i(\delta)u) f(x+y+ue_i)\,du\right|^p dx$$
$$\le\int_{G_{\varphi(\xi)}(\overline{x})}\left|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(t),x)\right)\Delta_i^{m_i}(\varphi_i(\delta)u) f(x+ue_i)\,du\right|^p dx$$
$$\le(\varphi_i(t))^{pl_i}\left\|\int_{-\infty}^{+\infty}\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(t),x)\right)\right.$$
$$\times\varphi_i(t)^{-l_i}\Delta_i^{m_i}\left(\varphi_i(\delta)u,G_{\varphi(t)}\right)fdu\|^p_{p,G_{\varphi(t)}(\overline{x})}$$

$$\le\varphi_i(t)^{p+pl_i}\left\|\varphi_i(t)^{-l_i}\Delta_i^{m_i}\left(\varphi_i(\delta),G_{\varphi(t)}\right)\right\|^p_{p,\varphi,\beta}\prod_{j=1}^n(\psi_j([\xi]_1))^{\beta_j p} \tag{1.16}$$

$$\int_{R^n}\left|M_i\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\rho'(\varphi(t),x)\right)\right|^s dy=\|M_i\|_s^s\cdot\prod_{j=1}\varphi_j(t). \tag{1.17}$$

From inequalities (1.14)-(1.17) it follows that

$$\|L_i(\cdot,t)\|_{rU_{\psi(\xi)}(\overline{x})}\le\|M_i\|_s\cdot\left\|(\varphi_i(t))^{-l_i}\Delta_i^{m_i}(\varphi_i(\delta))f\right\|_{p,\varphi,\beta;G}(\varphi_i(t))^{1+l_i}$$
$$\times\prod_{j=1}^n(\varphi_j(t))^{\frac{1}{s}+\beta_j p\left(\frac{1}{p}-\frac{1}{r}\right)}\cdot\prod_{j=1}^n(\psi_j([\xi]_1))^{\frac{\beta_j p}{r}}. \tag{1.18}$$

Inequality (1.11) is proved analogously. □

Inequalities (1.12), (1.13) and (1.18) for $r = q$ and for any $\overline{x} \in U$ reduce to the estimation

$$\left\|H_\eta^i\right\|_{rU_{\psi(\xi)}(\overline{x})} \le C_1 \left\|(\varphi_i(t))^{-l_i}\, \Delta_i^{m_i}\left(\varphi_i\left(\delta\right)\right) f\right\|_{p,\varphi,\beta;G} \times \left|Q_\eta^i\right| \prod_{j=1}^{n} \left(\psi_j([\xi]_1)\right)^{\beta_j \frac{p}{q}} \quad \left(Q_\eta^i < \infty\right). \tag{1.19}$$

In the case $Q_{\eta,T}^i < \infty$ inequality (1.10) can be proved in the same way.

From inequality (1.18) for $r = q$ and (1.19) we get the inequality ($\forall \overline{x} \in U$)

$$\sup_{\overline{x}\in U} \left\|L_i\right\|_{qU_{\psi(\xi)}(\overline{x})} \le C_2 \left\|(\varphi_i(t))^{-l_i}\, \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \cdot \prod_{j=1}^{n} \left(\psi_j([\xi]_1)\right)^{\beta_j \frac{p}{q}},$$

$$\sup_{\overline{x}\in U} \left\|H_\eta^i\right\|_{qU_{\psi(\xi)}(\overline{x})} \le C_3 \left\|(\varphi_i(t))^{-l_i}\, \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \cdot \prod_{j=1}^{n} \left(\psi_j([\xi]_1)\right)^{\beta_j \frac{p}{q}}.$$

From last inequalities it follows that

$$\left\|L_i\right\|_{q,\psi,\beta^1;U} \le C_1' \left\|(\varphi_i(t))^{-l_i}\, \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}, \tag{1.20}$$

$$\left\|H_\eta^i\right\|_{q,\psi,\beta^1;U} \le C_2' \left\|(\varphi_i(t))^{-l_i}\, \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}. \tag{1.21}$$

C_1' and C_2' are the constants independent of φ.

2 Main Results

Prove two theorems on the properties of the functions from the space $H_{p,\varphi,\beta}^l\left(G,\lambda\right)$.

Theorem 2.1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn, $1 \le p \le q \le \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \ge 0$ be entire $j = 1, 2, ..., n$, $Q_T^i < \infty$ $(i = 1, 2, ..., n)$ and let $f \in H_{p,\varphi,\beta}^l(G,\lambda)$. Then the following embeddings hold*

$$D^\nu : H_{p,\varphi,\beta}^l(G) \to L_{q,\psi,\beta^1}(G),$$

more precisely, for $f \in H_{p,\varphi,\beta}^l\left(G,\lambda\right)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are valid

$$\left\|D^\nu f\right\|_{q,G} \le C_1 \left(B(t)\|f\|_{q,\psi,\beta;G} + \sum_{i=1}^{n} \left|Q_T^i\right| \sup_{0<t<t_0} \left\|\frac{\Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f}{(\varphi_i(t))^{l_i}}\right\|_{p,\varphi,\beta;G}\right), \tag{2.1}$$

$$\left\|D^\nu f\right\|_{q,\psi,\beta^1;G} \le C_2 \left\|f\right\|_{H_{p,\varphi,\beta}^l(G,\lambda)},\ p \le q < \infty. \tag{2.2}$$

In particular, if

$$Q_{T,0}^i = \int_0^T \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_j p)\frac{1}{p}} \frac{\varphi_i'(t)}{\left(\varphi_i(t)\right)^{1-l_i}} dt < \infty, (i = 1, 2, \ldots, n), \tag{2.3}$$

then $D^{\nu} f(x)$ *is continuous on* G, *i.e*

$$\sup_{x\in G} |D^{\nu} f(x)| \le C_1 \Bigg(B(t) \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n} \left|Q^i_{T,0}\right| \sup_{0<t<t_0} \left\| \frac{\Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f}{\left(\varphi_i(t)\right)^{l_i}} \right\|_{p,\varphi,\beta;G} \Bigg) \tag{2.4}$$

$0 < T \le \min\{1, T_0\}$, T_0 *is a fixed number;* C_1, C_2, C_3, C_4 *are the constants independent of* f, *also* C_1 *and* C_3 *are independent from* T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^{\nu} f$ on G. Indeed, from the condition $Q^i_T < \infty$ for all $(i = 1, 2, ..., n)$ it follows that for $f \in H^l_{p,\varphi,\beta}(G) \to H^l_p(G)$, there exists $D^{\nu} f \in L_p(G)$ and for integral representation (1.3) and (1.4) with the same kernels is valid.

Applying the Minkowski inequality, from identities (1.3) and (1.4) we get

$$\|D^{\nu} f\|_{q,G} \le \left\| f^{(\nu)}_{\varphi(T)} \right\|_{q,G} + \sum_{i=1}^{n} \left\|H^i_T\right\|_{q,G}. \tag{2.5}$$

By means of inequality (1.11) for $U = G$, $M_i = K^i_i$, $t = T$ we get

$$\begin{aligned} \left\| f^{(\nu)}_{\varphi(t)} \right\|_{q,G} &\le \|f\|_{p,\varphi,\beta;G} \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}} \\ &\le C_1 A(t) \|f\|_{p,\varphi,\beta;G}, \end{aligned} \tag{2.6}$$

and by means of inequality (1.9) for $\eta = T$, $M_i = K^i_i$, $U = G$, we get

$$\left\|H^i_T\right\|_{q,G} \le C_2 Q^i_T \left\| (\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p,\varphi,\beta;G}. \tag{2.7}$$

Substituting (2.7) and (2.6) in (2.5), we get inequality (1.21). By means of inequalities (1.20) and (1.21) for $\eta = T$ we get inequality (2.2).

Now let conditions (2.3) be satisfied, then take into account identities (1.3), (1.4), from inequality (2.5) we get

$$\left\| D^{\nu} f - f^{(\nu)}_{\varphi(T)} \right\|_{\infty,G} \le C \sum_{i=1}^{n} \left|Q^i_T\right| \sup_{0<t<t_0} \left\| \frac{\Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f}{\left(\varphi_i(t)\right)^{l_i}} \right\|_{p,\varphi,\beta;G}.$$

As $T \to 0$, the left side of this inequality tends to zero, since $f^{(\nu)}_{\varphi(T)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^{\nu} f$ is continuous on G. □

Let γ be an n-dimensional vector.

Theorem 2.2. *Let all the conditions of Theorem 1 be fulfilled. Then for* $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ *the derivative* $D^{\nu} f$ *satisfies on* G *the Hölder generalized condition, i.e the following inequality is valid:*

$$\|\Delta(\gamma, G) D^{\nu} f\|_{q,G} \le C \|f\|_{H^l_{p,\varphi,\beta}(G)} \cdot |h(|\gamma|, \varphi; T)|, \tag{2.8}$$

where C *is a constant independent of* f, $|\gamma|$ *and* T.

In particular, if $Q^i_{T,0} < \infty$, $(i = 1, 2, \ldots, n)$, *then*

$$\sup_{x\in G} |\Delta(\gamma, G) D^{\nu} f(x)| \le C \|f\|_{H^l_{p,\varphi,\beta}(G_\varphi)} \cdot |h_0(|\gamma|, \varphi, T)|. \tag{2.9}$$

where $h(|\gamma|, \varphi, T) = \max\limits_i \left\{ |\gamma|, Q^i_{|\gamma|}, Q^i_{|\gamma|,T} \right\}$ $\left(h_0(|\gamma|, \varphi, T) = \max\limits_i \left\{ |\gamma|, Q^i_{|\gamma|,0}, Q^i_{|\gamma|,T,0} \right\} \right)$

Proof. According to Lemma 8.6 from [1] there exists a domain

$$G_\omega \subset G\,(\omega = \zeta r\,(x)\,,\ \zeta > 0\ r\,(x) = \rho\,(x, \partial G)\,,\ x \in G)$$

and assume that $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identities (1.3), (1.4) with the same kernels are valid. After the same transformations, from (1.3) and (1.4) we get

$$\begin{aligned}
&|\varDelta\,(\gamma, G)\,D^\nu f\,(x)| \le \prod_{j=1}^{n} (\varphi_j(t))^{-1-\nu_j}\\
&\times \int\limits_{R^n}\int\limits_{R^n} |f\,(x+y+z)| \left|\varOmega^{(\nu)}\left(\frac{y-\gamma}{\varphi(t)}, \frac{\rho\,(\varphi(t), x)}{2\varphi(t)}\right)\right.\\
&-\left.\varOmega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho\,(\varphi(t), x)}{2\varphi\,(T)}\right)\right| dydz\\
&+C_2\sum_{i=1}^{n}\left\{\int\limits_{0}^{|\gamma|}\int\limits_{R^n}\int\limits_{-\infty}^{+\infty}\left|\zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\,(\varphi_i\,(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho'\,(\varphi(t), x)\right)\right| \times -\right.\\
&\times\left|K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho\,(\varphi\,(t,x))}{\varphi(t)}\right)\right| |\varDelta_i^{m_i}\,(\varphi_i\,(\delta)\,u)\,f\,(x+y+ue_i)|\,dydudt\\
&+\int\limits_{|\gamma|}^{T}\int\limits_{R^n}\int\limits_{-\infty}^{+\infty}\left|K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho\,(\varphi\,(t,x))}{\varphi(t)}\right)\right|\left|\zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\,(\varphi_i\,(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho_i'\,(\varphi(t), x)\right)\right|\\
&\left.\times\int\limits_{0}^{1} |\varDelta_i^{m_i}\,(\varphi_i\,(\delta)\,u)\,f\,(x+y+v\gamma)|\,dvdudydt\right\}.\\
&= C_1 A\,(x,\gamma) + C_2\sum_{i=1}^{n}\left(E\,(x,\gamma) + F\,(x,\gamma)\right),
\end{aligned} \tag{2.10}$$

where $0 < T \le \{1, T_0\}$. Additionally, we assume that $|\gamma| < T$. Consequently, $|\gamma| < \min(\omega, T)$. If $x \in G \setminus G_\omega$, then

$$\varDelta\,(\gamma, G)\,D^\nu f\,(x) = 0.$$

By inequality (2.9) we have

$$\begin{aligned}
\|\varDelta\,(\gamma, G)\,D^\nu f\|_{q,G} \le{}& \|A\,(\cdot,\gamma)\|_{q,G_\omega}\\
&+\sum_{i=1}^{n}\left(\|E\,(\cdot,\gamma)\|_{q,G_\omega} + \|F\,(\cdot,\gamma)\|_{q,G_\omega}\right),
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
A\,(x,\gamma) \le{}& \prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-2}\int\limits_{0}^{|\gamma|} d\zeta \int\limits_{R^n}\int\limits_{R^n} |f\,(x+\zeta e_\gamma + y)|\\
&\times\left|D_j\varOmega^{(\nu)}\left(\frac{y}{\varphi\,(T)}, \frac{\rho\,(\varphi(t), x)}{2\varphi(t)}\right)\varOmega^{(\nu)}\left(\frac{z}{\varphi\,(T)}, \frac{\rho\,(\varphi(t), x)}{2\varphi(t)}\right)\right| dydz.
\end{aligned}$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, and applying the generalized Minkowski inequality, from inequality (1.11) for $U = G$, we have

$$\|A\,(\cdot,\gamma)\|_{q,G_\omega} \le C_1\,|\gamma|\,\|f\|_{p,\varphi,\beta;G}. \tag{2.12}$$

By means of inequality (1.9), for $U = G$, $\eta = |\gamma|$ we get

$$\|E\,(\cdot,\gamma)\|_{q,G_\omega} \le C_2\left|Q^i_{|\gamma|}\right|\left\|(\varphi_i(t))^{-l_i}\,\varDelta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \tag{2.13}$$

and by means of inequality (1.10) for $U = G$, $\eta = |\gamma|$ we get

$$\|F(\cdot,\gamma)\|_{q,G_\omega} \le C_3 \left|Q^i_{|\gamma|,T}\right| \left\|(\varphi_i(t))^{-l_i} \varDelta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}. \tag{2.14}$$

From inequalities (2.11) and (2.12)-(2.14) we get the required inequality.

Now suppose that $|\gamma| \ge \min(\omega, T)$. Then

$$\|\varDelta(\gamma, G) D^\nu f\|_{q,G} \le 2\|D^\nu f\|_{q,G} \le C(\omega T)\|D^\nu f\|_{q,G} |h(|\gamma|, \varphi; T)|.$$

Estimating for $\|D^\nu f\|_{q,G}$ by means of inequality (2.1), in this case we get estimation (2.8). □

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