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Least-Squares Method of EQ_1^{rot} Nonconforming Finite Element for Convection-Diffusion Problems

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Abstract In this paper, a least-squares method of EQ_1^{rot} nonconforming finite element(NFE) is proposed for convection-diffusion problems. The existence and uniqueness of the approximate solutions are proved. The convergence analysis is presented and the optimal order error estimates for the stress under $H(div)$-norm and the displacement under broken H^1-norm are derived. At last, some numerical results are presented to verify the theoretical analysis, which show that our method is stable and performs very well.

Keywords: convection-diffusion problems, EQ_1^{rot} NFE, least-squares method, optimal order error estimates.

1 Introduction

The convection-diffusion equation is a parabolic type partial differential equation. Its simplest form takes ([1]):

$$\begin{aligned} -k\triangle u + \mathbf{a}\cdot\nabla u = f \quad & in \quad \Omega, \\ u = g \quad & on \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$, $0 < k \leq 1$ is the constant viscosity coefficient (or diffusivity), $\mathbf{a} = (a_1, a_2)$ is the velocity field, f is a source function and g is a given boundary data. It is well-known that the solution u of problem (1.1) can exhibit localized phenomenon such as boundary and interior layers, i.e., narrow regions where the derivative of u is very large. This is the case for instance if the problem is convection-dominated, i.e., $0 < k \ll |\mathbf{a}|$. For simplifying the mathematical analysis, in what follows we assume that $\mathbf{a}$ is a constant vector with $|\mathbf{a}| = 1$ and $g = 0$ on the boundary $\partial\Omega$.

The convection-diffusion problems (CDPs) have a wide application in many mathematical environment studies to model pollutant transports in the atmosphere, groundwater and surface water. There has been significant interest in the design and analysis of numerical schemes for the CDPs such as the finite volume element methods [2, 3]; the mixed-hybrid FEM [4]; the Petrov-Galerkin methods [5, 6]; the least-squares mixed FEMs [1, 7] and the very popular streamline-diffusion FEMs [8, 9, 10]. In which, [8] presented the $O(h^{r+\frac{1}{2}})$ order error estimates in the L^2-norm on the triangular nonconforming element, which $r \geq 1$ is the order of complete polynomials contained in FE space; [9] handled mixed meshes consisting of triangles and quadrilaterals and gave the convergence analysis; [10] discussed multi-grid methods of the Q_1^{rot} NFE; [11] constructed P_1^{mod} NFE to approximate the CDPs on triangular meshes and presented error analysis. Recently, [12] applied the lowest order Crouzeix-Raviart NFE to approximate the CDPs and established a posteriori error estimator. [13, 14] studied low order nonconforming rectangular FEMs for the CDPs with a modified characteristic FE scheme and provided the $O(h^2)$ order error estimate in L^2-norm on anisotropic meshes. Moreover, [15] used conforming Q_1 element, P_1 element and EQ_1^{rot} NFE to approximate time-dependent advection-diffusion equations and got uniform error estimates. However, it seems that there are no reports on least-squares methods of NFEs for the CDPs.

As we know, the least-squares FEM [7] is not subject to the LBB condition and also always leads to symmetric system matrices, which implies that only half of the coefficients need to be stored. In addition, the system satisfies a priori coercivity inequality and the least-squares formulation generates positive definite algebraic system matrices, which can be solved using standard and robust iterative methods such as conjugate gradient methods. Recently, [16] proposed least-squares methods of NFEs for the second-order elliptic problem on different meshes in a unified way and gave the convergence analysis and error estimates, [17] studied the least-square Galerkin-Petrov method of NFE for the stationary conduction-convection problem and obtained the corresponding optimal order error estimates. So it is natural to ask whether this least-squares methods of NFEs can be applied to the CDPs or not. As the first attempt, we will give a positive answer in this paper.

The remainder of this paper is organized as follows: In next section, we will introduce the least-squares method of EQ_1^{rot} NFE for the CDPs. Section 3 will prove the existence and uniqueness of the discrete solutions and derive the corresponding optimal order error estimates. In the last section, we will provide some numerical results to confirm the theoretical analysis.

2 Least-Squares Method of EQ_1^{rot} NFE

Introducing the stress $\mathbf{p} = -k\nabla u$ in Ω as an independent variable, we can rewrite (1.1) in the first order system:

$$\begin{aligned} &\nabla\cdot\mathbf{p}+\mathbf{a}\cdot\nabla u=f \quad &in \quad &\Omega,\\ &\mathbf{p}+k\nabla u=0 \quad &in \quad &\Omega,\\ &u=0 \quad &on \quad &\partial\Omega. \end{aligned} \tag{2.1}$$

Let $X = \mathbf{H}(div;\Omega) = \{\mathbf{q} \in [L^2(\Omega)]^2, \nabla\cdot\mathbf{q} \in L^2(\Omega)\}$ equipped with the norm $\|\mathbf{q}\|_{div} = \|\mathbf{q}\| + \|\nabla\cdot\mathbf{q}\|$, where $\|\cdot\|$ is the L^2-norm.

A least-squares variational problem for (2.1) is to find $u \in U = H_0^1(\Omega)$ and $\mathbf{p} \in X$ such that

$$B(u,\mathbf{p};v,\mathbf{q}) = L(v,\mathbf{q}) \qquad \forall\ (v,\mathbf{q}) \in U\times X, \tag{2.2}$$

where the bilinear form $B(\cdot;\cdot)$ and the linear form $L(\cdot)$ are defined as

$$B(u,\mathbf{p};v,\mathbf{q}) = (\nabla\cdot\mathbf{p}+\mathbf{a}\cdot\nabla u, \nabla\cdot\mathbf{q}+\mathbf{a}\cdot\nabla v) + (\mathbf{p}+k\nabla u, \mathbf{q}+k\nabla v), \tag{2.3}$$

$$L(v,\mathbf{q}) = (f, \nabla\cdot\mathbf{q}+\mathbf{a}\cdot\nabla v). \tag{2.4}$$

The following lemma can be found in [1].

Lemma 1. *There exists a constant $C > 0$ such that for all $v \in U, \mathbf{q} \in X$,*

$$B(v,\mathbf{q};v,\mathbf{q}) \geq C(\|v\|_1^2 + \|\mathbf{q}\|_{div}^2),$$

here and later, C (with or without subscripts) is a generic constant, independent of h, which may take different values at different occurrences.

Thus, Lax-Milgram lemma guarantees that Problem (2.2) has a unique solution $(u,\mathbf{p}) \in U\times X$.

Let $\Gamma_h = \{K\}$ be a regular rectangular partition of Ω, $h_K = diam\{K\}$ and $h = \max_{K\in\Gamma_h}\{h_K\}$. We take the EQ_1^{rot} NFE space U_h (cf. [13, 15, 18, 19]) and zero order R-T element space X_h to approximate U and X, respectively.

Define the FE spaces U_h and X_h by

$$\begin{aligned} U_h &= \{v_h \in L^2(\Omega); v_h|_K \in P, \forall K \in \Gamma_h, \int_l [v_h]ds = 0, l \subset \partial K\},\\ X_h &= \{w \in X; w|_K \in Q_{1,0}(K)\times Q_{0,1}(K), \forall K \in \Gamma_h\}, \end{aligned}$$

where $P = span\{1, x, y, x^2, y^2\}$, $[v_h]$ denotes the jump of v_h across the boundary l of K if l is an internal edge, and $[v_h] = v_h$ if $l \subset \partial\Omega$.

Let $I_h : H^1(\Omega) \to U_h$ and $\mathbf{\Pi}_h : (H^1(\Omega))^2 \to X_h$ be the associated interpolation operators satisfying $I_h|_K = I_K, \mathbf{\Pi}_h|_K = \mathbf{\Pi}_K$, then we have

$$\int_K (v - I_K v)dxdy = 0, \int_{l_i}(v - I_K v)ds = 0, \int_{l_i}(\mathbf{q} - \mathbf{\Pi}_K\mathbf{q})\cdot\mathbf{n}_i ds = 0,$$

where l_1, l_2, l_3, l_4 are four edges of ∂K, $\mathbf{n}_i$ is the unit outward normal vector to $l_i (i = 1, 2, 3, 4)$.

For $v_h \in U_h$, it has been proved in [15] and [20] that the following conclusions, i.e.,

$$(\mathbf{a} \cdot \nabla_h v_h, v_h) = 0 \tag{2.5}$$

and

$$\|v_h\| \leq C_1 \|v_h\|_h \tag{2.6}$$

hold, respectively. Where ∇_h is the gradient operator defined element by element, $\|\cdot\|_h = \sqrt{\sum_K |\cdot|^2_{1,K}}$ is a norm on U_h.

Applying Green's formula, we can derive for $(v_h, \mathbf{q}_h) \in U_h \times X_h$

$$\sum_K \int_{\partial K} \mathbf{q}_h \cdot \mathbf{n}_K v_h ds = 0, \tag{2.7}$$

where $\mathbf{n}_K$ is the unit outward normal vector to ∂K.

The least-squares scheme for Problem (2.2) is to find $(u_h, \mathbf{p_h}) \in U_h \times X_h$ such that

$$B_h(u_h, \mathbf{p_h}; v_h, \mathbf{q_h}) = L(v_h, \mathbf{q_h}) \quad \forall (v_h, \mathbf{q_h}) \in U_h \times X_h, \tag{2.8}$$

where

$$\begin{aligned} B_h(u_h, \mathbf{p_h}; v_h, \mathbf{q_h}) = & (\nabla \cdot \mathbf{p_h} + \mathbf{a} \cdot \nabla_h u_h, \nabla \cdot \mathbf{q_h} + \mathbf{a} \cdot \nabla_h v_h) \\ & + (\mathbf{p_h} + k \nabla_h u_h, \mathbf{q_h} + k \nabla_h v_h), \end{aligned} \tag{2.9}$$

$$L(v_h, \mathbf{q_h}) = (f, \nabla \cdot \mathbf{q_h} + \mathbf{a} \cdot \nabla_h v_h). \tag{2.10}$$

3 Solvability of the Discrete Problem and Error Estimates

In this section, we will prove the solvability of Problem (2.8) and give error estimates. The following theorem guarantees that Problem (2.8) has a unique solution.

Theorem 1. *For $(v_h, \mathbf{q_h}) \in U_h \times X_h$, there exist two positive constants C_2, C_3 such that*

$$C_2 \{\|v_h\|_h^2 + \|\mathbf{q_h}\|^2_{div}\} \leq B_h(v_h, \mathbf{q_h}; v_h, \mathbf{q_h}) \leq C_3 \{\|v_h\|_h^2 + \|\mathbf{q_h}\|^2_{div}\}. \tag{3.1}$$

Proof. The right hand of (3.1) is obvious. We proceed to show the left hand of (3.1).

Let α be a positive constant that will be determined later. We have

$$\begin{aligned} l.h.s. = & \|\nabla \cdot \mathbf{q}_h + \mathbf{a} \cdot \nabla_h v_h - \alpha v_h\|^2 + 2\alpha(\nabla \cdot \mathbf{q}_h + \mathbf{a} \cdot \nabla_h v_h, v_h) - \alpha^2 \|v_h\|^2 \\ & + \|\mathbf{q}_h + k \nabla_h v_h - \alpha \nabla_h v_h\|^2 + 2\alpha(\mathbf{q}_h + k \nabla_h v_h, \nabla_h v_h) - \alpha^2 \|\nabla_h v_h\|^2. \end{aligned}$$

In fact, by (2.7), we obtain$(\nabla \cdot \mathbf{q_h}, v_h) = -(\mathbf{q_h}, \nabla_h v_h)$.

So, from (2.5),(2.6), and choosing $\alpha = k/(1 + C_1^2)$, we get

$$\begin{aligned} l.h.s. & = (2\alpha k - \alpha^2) \|\nabla_h v_h\|^2 - \alpha^2 \|v_h\|^2 \\ & \geq (2\alpha k - \alpha^2 - C_1^2 \alpha^2) \|\nabla v_h\|^2 \\ & = (k^2/(1 + C_1^2)) \|v_h\|_h^2. \end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned} \|\mathbf{q_h}\|^2 & \leq 2(\|\mathbf{q_h} + k \nabla_h v_h\|^2 + k^2 \|\nabla_h v_h\|^2) \\ & \leq 2(B_h(v_h, \mathbf{q_h}; v_h, \mathbf{q_h}) + (1 + C_1^2) B_h(v_h, \mathbf{q_h}; v_h, \mathbf{q_h})) \\ & = 2(2 + C_1^2) B_h(v_h, \mathbf{q_h}; v_h, \mathbf{q_h}), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \|\nabla \cdot \mathbf{q_h}\|^2 & \leq 2\|\nabla \cdot \mathbf{q}_h + \mathbf{a} \cdot \nabla_h v_h\|^2 + 2\|\mathbf{a} \cdot \nabla_h v_h\|^2 \\ & \leq 2B_h(v_h, \mathbf{q_h}; v_h, \mathbf{q_h}) + C\|\nabla_h v_h\|^2 \\ & \leq (C/k^2) B_h(v_h, \mathbf{q_h}; v_h, \mathbf{q_h}). \end{aligned} \tag{3.4}$$

Combining (3.2)-(3.4) yields the desired result. □

Theorem 2. *Let* $(u, \mathbf{p}) \in (U \cap H^2(\Omega)) \times (X \cap \mathbf{H}^1(div; \Omega))$ *and* $(u_h, \mathbf{p_h}) \in U_h \times X_h$ *be the solutions of Problems (2.2) and (2.8), respectively. Then*

$$\|u - u_h\|_h + \|\mathbf{p} - \mathbf{p_h}\|_{div} \le Ch\{|u|_2 + |\nabla \cdot \mathbf{p}|_1 + |\mathbf{p}|_1\}. \tag{3.5}$$

Proof. For $v_h \in U_h, \mathbf{q_h} \in X_h$, we have

$$\begin{aligned} \|v_h - u_h\|_h^2 + \|\mathbf{q_h} - \mathbf{p_h}\|_{div}^2 &\le B_h(v_h - u_h, \mathbf{q_h} - \mathbf{p_h}; v_h - u_h, \mathbf{q_h} - \mathbf{p_h}) \\ &\le B_h(v_h - u, \mathbf{q_h} - \mathbf{p}; v_h - u_h, \mathbf{q_h} - \mathbf{p_h}) \\ &+ B_h(u - u_h, \mathbf{p} - \mathbf{p_h}; v_h - u_h, \mathbf{q_h} - \mathbf{p_h}). \end{aligned} \tag{3.6}$$

From (2.2)-(2.4) and (2.8)-(2.10), we find

$$B_h(u, \mathbf{p}; v_h - u_h, \mathbf{q_h} - \mathbf{p_h}) = B_h(u_h, \mathbf{p_h}; v_h - u_h, \mathbf{q_h} - \mathbf{p_h}). \tag{3.7}$$

Therefore,

$$\|u - u_h\|_h + \|\mathbf{p} - \mathbf{p_h}\|_{div} \le C\{\|u - I_h u\|_h + \|\mathbf{p} - \mathbf{\Pi}_h \mathbf{p}\|_{div}\}. \tag{3.8}$$

On the other hand, for each $K \in \Gamma_h$

$$\begin{aligned} \nabla \cdot \mathbf{\Pi}_h \mathbf{p}|_K &= \frac{1}{|K|}\int_K \nabla \cdot \mathbf{\Pi}_h \mathbf{p} dxdy = \frac{1}{|K|}\int_{\partial K} \mathbf{\Pi}_h \mathbf{p} \cdot \mathbf{n}_K ds \\ &= \frac{1}{|K|}\int_{\partial K} \mathbf{p} \cdot \mathbf{n}_K ds = \frac{1}{|K|}\int_K \nabla \cdot \mathbf{p} dxdy. \end{aligned} \tag{3.9}$$

Thus we have $\nabla \cdot \mathbf{\Pi}_h|_K = P_0 \nabla \cdot |_K$, where P_0 is the local L^2 projection satisfying

$$\|\nabla \cdot (\mathbf{p} - \mathbf{\Pi}_h \mathbf{p})\|_{0,K} = \|\nabla \cdot \mathbf{p} - P_0 \nabla \cdot \mathbf{p}\|_{0,K} \le Ch|\nabla \cdot \mathbf{p}|_{1,K}. \tag{3.10}$$

Substituting (3.10) into (3.8) and applying the interpolation theory yield the desired result. □

Remark 1. *We point out that (2.5) and (2.7) are the key conditions leading to the optimal order error estimates, and the results obtained in the present work are also valid if* U_h *is replaced by the* Q_1^{rot} *NFE space discussed in [14, 20] on the square meshes, the constrained* Q_1^{rot} *NFE and* P_1 *NFE spaces proposed in [21] and [22] on the rectangular meshes, respectively. On the other hand, it can be checked that if* U_h *is not changed but we replace the FE space* X_h *with that of [23, 24], then the above results are also valid.*

4 Numerical Results

In this section, we present some numerical results to confirm our theoretical analysis.

We consider the convection-diffusion equation (1.1) with homogeneous Dirichlet boundary condition. The computational domain is set as $\Omega = [0,1] \times [0,1]$ and triangulated by uniform rectangular meshes with N nodes. we consider the following smooth solution of (1.1)

$$u = \sin(\pi x)\sin(\pi y),$$

with $\mathbf{a} = (1,1)$ and different values of k.

The error estimates of u under broken H^1-norm and $\mathbf{p}$ under $H(div)$-norm are displayed in the following Tables 1-4 for different k. We can observe first order convergence for both u and $\mathbf{p}$, which is consistent with the theoretical analysis.

Numerical experiments show that when $k = 1.0e-1, 1.0e-2, 1.0e-3, 1.0e-4$, the figures of the FE solutions u_h are very nearly the same, so we only plot the figures of the exact solution u and the FE solution u_h for $k = 1.0e-4$ (see Figure 1 and Figure 2).

Table 1. Error estimates of u and $\mathbf{p}$ for $k = 1.0e-1$.

N	$\|u-u_h\|_h$	rate	$\|\mathbf{p}-\mathbf{p}_h\|_{div}$	rate
25	0.764634794711	\	0.336452643135	\
81	0.386294524031	0.9851	0.162609603986	1.0490
289	0.193591337743	0.9967	0.080427574859	1.0157
1089	0.096848601125	0.9992	0.040095337767	1.0043
4225	0.048430824692	0.9998	0.020032547836	1.0011

Table 2. Error estimates of u and $\mathbf{p}$ for $k = 1.0e-2$.

N	$\|u-u_h\|_h$	rate	$\|\mathbf{p}-\mathbf{p}_h\|_{div}$	rate
25	0.697902874070	\	0.034608162349	\
81	0.354583797115	0.9769	0.016626320708	1.0576
289	0.178002130926	0.9942	0.008148952342	1.0288
1089	0.089089309192	0.9986	0.004033799337	1.0145
4225	0.044555501545	0.9996	0.002007374401	1.0068

Table 3. Error estimates of u and $\mathbf{p}$ for $k = 1.0e-3$.

N	$\|u-u_h\|_h$	rate	$\|\mathbf{p}-\mathbf{p}_h\|_{div}$	rate
25	0.697139928223	\	0.003461376108	\
81	0.354228960570	0.9768	0.001663397639	1.0572
289	0.177831372720	0.9942	0.000815735605	1.0280
1089	0.089005591104	0.9985	0.000404148478	1.0132
4225	0.044514041147	0.9996	0.000201165557	1.0065

Table 4. Error estimates of u and $\mathbf{p}$ for $k = 1.0e-4$.

N	$\|u-u_h\|_h$	rate	$\|\mathbf{p}-\mathbf{p}_h\|_{div}$	rate
25	0.697132295521	\	0.000346138085	\
81	0.354225408909	0.9768	0.000166340398	1.0572
289	0.177829660595	0.9942	0.000081574282	1.0280
1089	0.089004750140	0.9985	0.000040415614	1.0132
4225	0.044513624478	0.9996	0.000020117642	1.0065

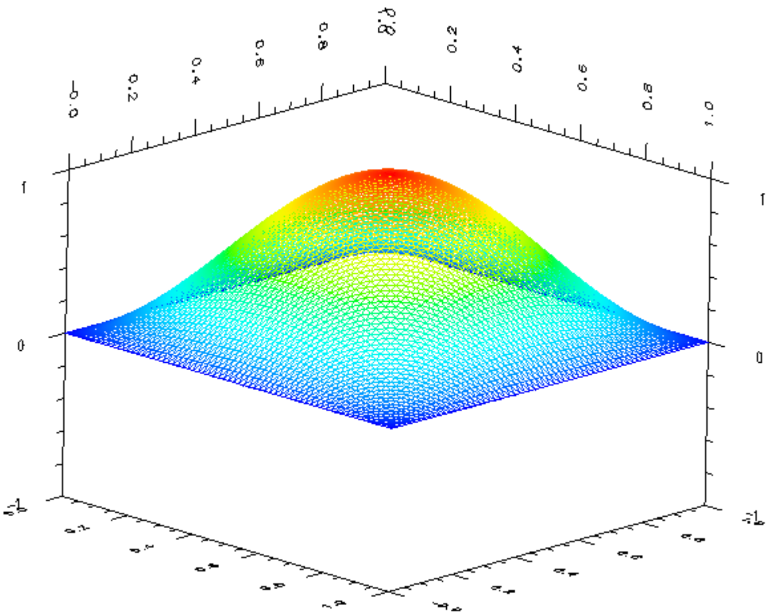

Figure 1. The plot of the exact solution u.

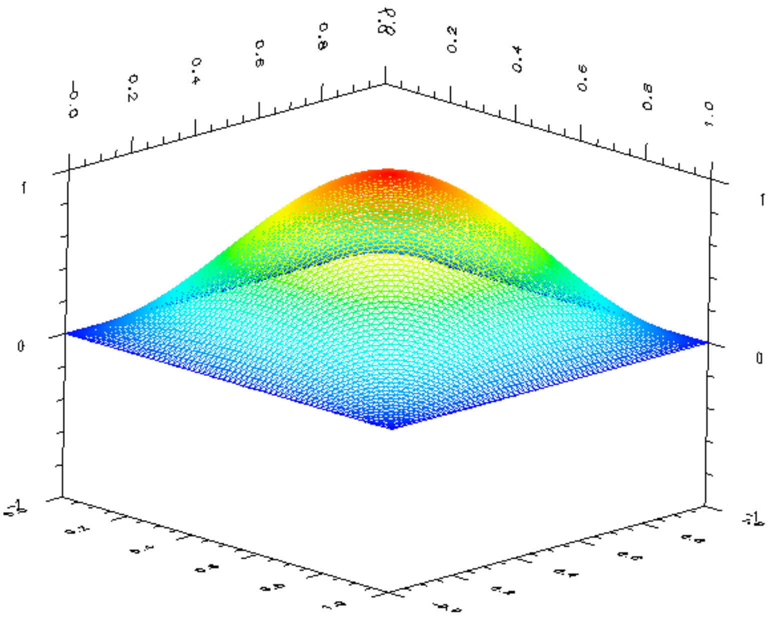

Figure 2. The plot of the FE solution u_h for with $k = 1.0e - 4$.

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