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Two Efficient Bi-Parametric Derivative Free With Memory Methods for Finding Simple Roots Nonlinear Equations

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Abstract The present paper is devoted to the improvement of the existing fourth-and eighth-order derivative free methods without memory proposed by Cordero et al. (2013). To achieve this goal two parameters are introduced which are calculated with the help of Newton's interpolatory polynomial. It is shown that the R-order convergence of the proposed methods has been increased from 4 to 7 and 8 to 14, respectively without any extra evaluation.Two non-smooth examples are demonstrated to confirm theoretical results. Numerically the modified methods are examined along with comparison to recent existing with memory methods.

Keywords: Derivative free method, nonlinear equations, order of convergence, efficiency index, nonsmooth function.

1 Introduction

The main motive in constructing iterative algorithms for solving nonlinear equations is to achieve as high as possible convergence rate with a fixed number of function evaluations per iteration. In this work we study the multipoint methods with memory, a work which is very rarely discussed in the literature in spite of high computational efficiency of this kind of root-finding methods. Most of these methods are improvement of multipoint methods without memory with optimal order of convergence. Using Newton's interpolation with divided difference, deadly fast convergence of new methods with memory is attained without adding more function evaluations. As a result, these multipoint methods hold a very high computational efficiency. Let p_k represent the $m+1$ quantities $x_k, t_1(x_k), t_2(x_k), ..., t_m(x_k)$, $k \geq 1$ and define an iterative process by

$$x_{k+1} = F(p_k; p_{k-1}, p_{k-2}, ..., p_{k-m}).$$

Following Traub's terminology [10], F is called a multipoint iteration function with memory. Probably Traub initiated the idea of with memory method in his book [10]. For this he considered Steffensen type method

$$x_{k+1} = x_k - \frac{\gamma f(x_k)^2}{f(x_k + \gamma f(x_k)) - f(x_k)}, \tag{1}$$

where γ is arbitrary parameter. This method has quadratic convergence. To compare iterative methods theoretically, Owtrowski [1] introduced the idea of efficiency index given by r^{1/θ_f}, where r is the order of convergence and θ_f number of function evaluations per iteration. In other words we can say that an iterative method with higher efficiency index is more efficient. To accelerate the convergence order of this method without using additional evaluation γ is recursively calculated by self-accelerating method. Let γ_0 be the given initial parameter and consider

$$\begin{aligned} \phi_k &= \frac{f(x_k + \gamma_k f(x_k)) - f(x_k)}{\gamma_k f(x_k)},\ k = 0, 1, 2, ..., \\ x_{k+1} &= x_k - \frac{f(x_k)}{\phi_k}. \end{aligned} \tag{2}$$

where

$$\gamma_k = -\frac{1}{\phi_{k-1}}, \; k = 1, 2, ... \tag{3}$$

Traub derived that order of convergence of this method is 2.414. And thus the order of convergence of (2) with memory is more than that of Steffensen method, which also needs two function evaluations per iteration. Motivated by this currently researchers are trying to increase the efficiency of the existing optimal order without memory methods by using single or double parameters. In the literature with memory methods with two parameters are very rare.

In the present paper we present an improvement of the existing optimal fourth-and eighth-order derivative free method constructed by introducing two self accelerating parameters. These parameters are calculated with the help of Newton's interpolatory polynomial. In section 2, derivative free two- and three-points methods with memory with improved order of convergence from 4 to 7 and 8 to 14, respectively without extra evaluations are presented. Two non-smooth equations have been considered to give the comparisons of absolute errors and computational efficiencies are given in section 3 to illustrate convergence behavior. Finally, we give the concluding remark.

2 Development and Construction of With Memory Method

In the convergence analysis of the new method, we employ the notation used in Traub's book [10]: if m_k and n_k are null sequences and $m_k/n_k \to C$, where C is a non-zero constant, we shall write $m_k = O(n_k)$ or $m_k \sim Cn_k$. We also use the concept of R-order of convergence introduced by Ortega and Rheinboldt [11]. Let x_k be a sequence of approximations generated by an iterative method (IM). If this sequence converges to a zero ξ of function f with the R-order $O_R((IM), \xi) \geq r$, we will write

$$e_{k+1} \sim A_{k,r} e_k^r,$$

where $A_{k,r}$ tends to the asymptotic error constant A_r of the iterative method (IM) when $k \to \infty$.

Very recently Cordero et al. [2] presented derivative-free optimal fourth- and eighth-order iterative methods as follows:
For given x_0, consider

$$\begin{aligned} z_n &= x_n + f(x_n), \; n = 0, 1, 2, ..., \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, z_n]}, \\ x_{n+1} &= y_n - \frac{f(y_n) f[x_n, z_n]}{f[x_n, y_n] f[y_n, z_n]} \end{aligned} \tag{4}$$

and

$$\begin{aligned} z_n &= x_n + f(x_n), \; n = 0, 1, 2, ..., \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, z_n]}, \\ u_n &= y_n - \frac{f(y_n) f[x_n, z_n]}{f[x_n, y_n] f[y_n, z_n]}, \\ x_{n+1} &= u_n - \frac{f(u_n)}{b_2 - b_1 b_4}, \end{aligned} \tag{5}$$

where

$$b_4 = \frac{f[y_n, u_n, x_n] - f[y_n, u_n, z_n]}{f[y_n, z_n] - f[y_n, x_n]},$$

$$b_3 = f[y_n, u_n, z_n] + b_4 f[y_n, z_n],$$

$$b_2 = f[y_n, u_n] - b_3(y_n - u_n) + f(y_n)b_4,$$

$$b_1 = f(u_n).$$

If we introduce two different parameters in z_n and y_n involved in the above methods then the modified methods are given by along with its error expressions as follows:
Modified method I. For suitably given x_0,

$$\begin{aligned} z_n &= x_n + \epsilon f(x_n),\ n = 0, 1, 2, ..., \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, z_n] + \delta f(z_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)f[x_n, z_n]}{f[x_n, y_n]f[y_n, z_n]}, \end{aligned} \tag{6}$$

and its error expression is given by

$$e_{n+1} = \frac{(1 + \epsilon c_1)^2(\delta c_1 + c_2)(2c_2^2 + c_1(\delta c_2 - c_3))}{c_1^3} e_n^4 + O(e_n^5). \tag{7}$$

Modified method II. For suitably given x_0,

$$\begin{aligned} z_n &= x_n + \alpha f(x_n),\ n = 0, 1, 2, ..., \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, z_n] + \beta f(z_n)}, \\ u_n &= y_n - \frac{f(y_n)f[x_n, z_n]}{f[x_n, y_n]f[y_n, z_n]}, \\ x_{n+1} &= u_n - \frac{f(u_n)}{b_2 - b_1 b_4}, \end{aligned} \tag{8}$$

and its error expression is given by

$$\begin{aligned} e_{n+1} = &\frac{(1 + \alpha c_1)^4(\beta c_1 + c_2)^2(2c_2^2 + c_1(\beta c_2 - c_3))(2c_2^4 + c_1 c_2^2(\beta c_2 - c_3) - c_1^2 c_3^2 + c_1^2 c_2 c_4)}{c_1^7 c_2} e_n^8, \\ &+ O(e_n^9), \end{aligned} \tag{9}$$

where b_1, b_2, b_3, b_4 are as previously defined and $c_i = \frac{f^{(i)}(\xi)}{i!}$. Since the above error equations contain the parameters, which can be approximated in such a way that increases the local convergence order. For this purpose, first we put $\epsilon = \epsilon_k$, $\delta = \delta_k$ and $\alpha = \alpha_k$, $\beta = \beta_k$ and then approximations of these parameters are given by

$$\begin{aligned} \epsilon_k &= -\frac{1}{c_1} \approx -\frac{1}{\tilde{c_1}} = -\frac{1}{N_3'(x_k)}, \\ \delta_k &= -\frac{c_2}{c_1} \approx -\frac{\overline{c_2}}{\overline{c_1}} = -\frac{N_4''(w_k)}{2N_4'(w_k)}. \end{aligned} \tag{10}$$

and

$$\begin{aligned} \alpha_k &= -\frac{1}{c_1} \approx -\frac{1}{\tilde{c_1}} = -\frac{1}{\tilde{N}_4'(x_k)}, \\ \beta_k &= -\frac{c_2}{c_1} \approx -\frac{\overbrace{c_2}}{\overbrace{c_1}} = -\frac{\tilde{N}_5''(w_k)}{2\tilde{N}_5'(w_k)}. \end{aligned} \tag{11}$$

where
$N_3(t) = N_3(t; x_k, y_{k-1}, x_{k-1}, z_{k-1})$, $N_4(t) = N_4(t; x_k, w_k, y_{k-1}, x_{k-1}, z_{k-1})$ and
$\tilde{N}_4(t) = \tilde{N}_4(t; x_k, u_{k-1}, y_{k-1}, x_{k-1}, z_{k-1})$, $\tilde{N}_5(t) = \tilde{N}_5(t; x_k, w_k, u_{k-1}, y_{k-1}, x_{k-1}, z_{k-1})$
are the Newton's interpolatory polynomial of degree three, four and five respectively. Before going to prove the main result, we state the following two lemmas which can be obtained by using the error of Newton's interpolation, in the same manner as in [3].

Lemma 1. If $\epsilon_k = -\frac{1}{N_3'(x_k)}$ and $\delta_k = -\frac{N_4''(w_k)}{2N_4'(w_k)}$, then the estimates

$$(i)\ 1 + \epsilon_k c_1 \sim \frac{c_4}{c_1} e_{k-1,y} e_{k-1,z} e_{k-1},$$
$$(ii)\ \delta_k c_1 + c_2 \sim -c_5 e_{k-1,y} e_{k-1,z} e_{k-1}.$$

Lemma 2. If $\alpha_k = -\frac{1}{\tilde{N}_4'(x_k)}$ and $\beta_k = -\frac{\tilde{N}_5''(w_k)}{2\tilde{N}_5'(w_k)}$, then the estimates

$$(i)\ 1 + \alpha_k c_1 \sim -\frac{c_5}{c_1} e_{k-1,u} e_{k-1,y} e_{k-1,z} e_{k-1},$$
$$(ii)\ \beta_k c_1 + c_2 \sim c_6 e_{k-1,u} e_{k-1,y} e_{k-1,z} e_{k-1}.$$

The theoretical proof of the order of convergence of the proposed methods is given by the following theorem:

Theorem 1. If an initial approximation x_0 is sufficiently close to a simple zero ξ of $f(x)$ and the parameters ϵ_k, δ_k and α_k, β_k in the iterative scheme (6) and (8) are recursively calculated by the forms given in (10) and (11), respectively. Then the R-order of convergence of with memory schemes (6) and (8) is at least seven and fourteen, respectively.

Proof. First, we assume that the R-order of convergence of the sequence x_k, z_k, y_k, u_k is at least m, m_1, m_2 and m_3, respectively. Hence

$$e_{k+1} \sim A_{k,m} e_k^m \sim A_{k,m}(A_{k-1,m} e_{k-1}^m)^m \sim A_{k,m} A_{k-1,m}^m e_{k-1}^{m^2}. \tag{12}$$

and

$$e_{k,z} \sim A_{k,m_1} e_k^{m_1} \sim A_{k,m_1}(A_{k-1,m} e_{k-1}^m)^{m_1} \sim A_{k,m_1} A_{k-1,m}^{m_1} e_{k-1}^{mm_1}. \tag{13}$$

Similarly

$$e_{k,y} \sim A_{k,m_2} A_{k-1,m}^{m_2} e_{k-1}^{mm_2}, \tag{14}$$

$$e_{k,u} \sim A_{k,m_3} A_{k-1,m}^{m_3} e_{k-1}^{mm_3}. \tag{15}$$

Now we will prove the results in two parts. First for method (6) and then for (8).

Modified method I. For method (6), it can be derived that

$$e_{k,z} \sim (1 + \epsilon_k c_1) e_k, \tag{16}$$

$$e_{k,y} \sim L_1 (1 + \epsilon_k c_1)(\delta_k c_1 + c_2) e_k^2, where\ L_1 = \frac{1}{c_1}, \tag{17}$$

$$e_{k+1} \sim L_2 (1 + \epsilon_k c_1)^2 (\delta_k c_1 + c_2) e_k^4, where\ L_2 = \frac{c_2(2c_2^2 + c_1(\delta_k c_2 - c_3))}{c_1^3}. \tag{18}$$

Using the results of lemma (2.1) in the equations (16), (17) and (18), we have

$$e_{k,z} \sim \frac{c_4}{c_1}(A_{k-1,m_2})(A_{k-1,m_1})(A_{k-1,m}) e_{k-1}^{m_2+m_1+m+1}, \tag{19}$$

$$e_{k,y} \sim -\frac{c_4 c_5}{c_1} L_1 (A_{k-1,m_2}^2)(A_{k-1,m_1}^2)(A_{k-1m}^2) e_{k-1}^{2m_2+2m_1+2m+2}, \tag{20}$$

and

$$e_{k+1} \sim -\frac{c_4^2 c_5}{c_1^2} L_2 (A^3_{k-1,m_2})(A^3_{k-1,m_1})(A^4_{k-1,m}) e_{k-1}^{3m_2+3m_1+4m+3}. \tag{21}$$

Now comparing the equal powers of e_{k-1} in (13)-(19); (14)- (20) and (12)- (21), we get the following nonlinear system

$$\begin{aligned} mm_1 - m_2 - m_1 - m - 1 &= 0, \\ mm_2 - 2m_2 - 2m_1 - 2m - 2 &= 0, \\ m^2 - 3m_2 - 3m_1 - 4m - 3 &= 0. \end{aligned}$$

After solving these equations, we get $m = 7$, $m_2 = 4$, $m_1 = 2$. It confirms the convergence of method (6). This shows the first part.
Modified method II. For method (8), it can be derived that

$$e_{k,z} \sim (1 + \alpha_k c_1) e_k, \tag{22}$$

$$e_{k,y} \sim L_1 (1 + \alpha_k c_1)(\beta_k c_1 + c_2) e_k^2, where\ L_1 = \frac{1}{c_1}, \tag{23}$$

$$e_{k,u} \sim O_1 (1 + \alpha_k c_1)^2 (\beta_k c_1 + c_2) e_k^4, \tag{24}$$

where $O_1 = \frac{c_2(2c_2^2 + c_1(\beta_k c_2 - c_3))}{c_1^3}$ and

$$e_{k+1} \sim O_2 (1 + \alpha_k c_1)^4 (\beta_k c_1 + c_2)^2 e_k^8, \tag{25}$$

where $O_2 = \frac{(2c_2^2 + c_1(\beta_k c_2 - c_3))(2c_2^4 + c_1 c_1^2(\beta_k c_2 - c_3) + c_1^2(-c_3^2 + c_2 c_4))}{c_1^7 c_2}$. Now using the results of lemma (2.2) in the equations (22), (23), (24) and (25), we have

$$e_{k,z} \sim -\frac{c_5}{c_1}(A_{k-1,m_3})(A_{k-1,m_2})(A_{k-1,m_1})(A_{k-1,m}) e_{k-1}^{m_3+m_2+m_1+m+1}, \tag{26}$$

$$e_{k,y} \sim -\frac{c_5 c_6}{c_1} L_1 (A^2_{k-1,m_3})(A^2_{k-1,m_2})(A^2_{k-1,m_1})(A^2_{k-1m}) e_{k-1}^{2m_3+2m_2+2m_1+2m+2}, \tag{27}$$

$$e_{k,u} \sim \left(\frac{c_5^2 c_6}{c_1^2}\right) O_1 (A^3_{k-1,m_3})(A^3_{k-1,m_2})(A^3_{k-1,m_1})(A^4_{k-1,m}) e_{k-1}^{3m_3+3m_2+3m_1+4m+3}. \tag{28}$$

and

$$e_{k+1} \sim \left(\frac{c_5^4 c_6^2}{c_1^4}\right) O_2 (A^6_{k-1,m_3})(A^6_{k-1,m_2})(A^6_{k-1,m_1})(A^8_{k-1,m}) e_{k-1}^{6m_3+6m_2+6m_1+8m+6}. \tag{29}$$

Comparing the equal powers of e_{k-1} in (13)-(26); (14)- (27); (16)- (28) and (12)- (29), we get the following nonlinear system

$$\begin{aligned} mm_1 - m_3 - m_2 - m_1 - m - 1 &= 0, \\ mm_2 - 2m_3 - 2m_2 - 2m_1 - 2m - 2 &= 0, \\ mm_3 - 3m_3 - 3m_2 - 3m_1 - 4m - 3 &= 0, \\ m^2 - 6m_3 - 6m_2 - 6m_1 - 8m - 6 &= 0. \end{aligned}$$

After solving these equations we get $m = 14$, $m_3 = 7$, $m_2 = 4$, $m_1 = 2$. And thus proof is completed.

Note 1.: The efficiency index of the proposed method (6) along with (10) is $7^{1/3} = 1.9129$ which is more than $4^{1/3} = 1.5874$ of method (4).
Note 2.: The efficiency index of the proposed method (8) along with (11) is $14^{1/4} = 1.9343$ which is more than $8^{1/4} = 1.6818$ of method (5).

3 Application to Non-smooth Equations

In this section we are going to check the effectiveness of the new with memory methods (6) with (10) (NMWM7) and (8) with (12) (NMWM14), comparing them with some recent established with memory methods. Specifically, we consider the sixth-order method (JMWM6) introduced by Jovana in [3], sixth-order method (LMWM6) introduced by Lotfi et al. in [4], seventh-order method (CMWM7) introduced by Cordero et al. in [5], twelfth-order method I (LTMWM12I), II (LTMWM12II), III (LTMWM12III) and IV (LTMWM12IV) introduced by Lotfi and Tavakoli in [6], twelfth-order method I (EMWM12I), II (EMWM12II) and III (EMWM12III) introduced by Eftekhari in [7] and fourteenth-order I (LMWM14I) and II (LMWM14II) introduced by Lotfi et al. in [8]. Nowadays, high-order methods are important because numerical applications use high precision in their computations; for this reason numerical tests have been carried out using variable precision arithmetic in MATHEMATICA 8 with 100 significant digits. Tables 1 and 2 show the absolute error for the first, second and third iterations. To check the theoretical order of convergence, we calculate the computational order of convergence (COC) using the following formula:

$$COC = \frac{ln(|f(x_n)/f(x_{n-1})|)}{ln(|f(x_{n-1})/f(x_{n-2})|)}.$$

We test the performances of new methods for the following two non-smooth functions:

$$1.\ f_1(x) = 10(x^4 + x),\ x < 0$$
$$= -10(x^3 + x),\ x \geq 0.$$

$$2.\ f_2(x) = x(x + 1),\ x < 0$$
$$= -2x(x - 1),\ x \geq 0.$$

Table 1. Numerical results for $f_1(x)$.

Method	$\lvert x_1 - \xi\rvert$	$\lvert x_2 - \xi\rvert$	$\lvert x_3 - \xi\rvert$	COC
	$x_0 = -0.8,$	$\gamma_0 = 0.01,$	$\alpha_0 = 0.01,$	$\xi = -1$
JMWM6	0.15732e+0	0.36669e-5	0.34403e-37	6.7250
LMWM6	0.16036e+0	0.25566e-3	0.15859e-19	5.5324
CMWM7	0.14660e-1	0.10220e-11	0.62905e-83	7.0025
NMWM7	0.49246e-1	0.78791e-8	0.30306e-55	6.9341
LTMWM12I	0.18233e+1	0.99985e+0	1.00000e+0	2.2130
LTMWM12II	0.86418e-1	0.52042e-9	0.00000e+0	11.824
LTMWM12III	0.12019e-1	0.36933e-6	0.00000e+0	12.343
LTMWM12IV	0.36954e+0	0.36954e+0	0.36954e+0	1.0042
EMWM12I	0.57265e-1	0.10604e-9	0.00000e+0	11.117
EMWM12II	0.14034e+2	0.47011e+1	0.11965e+1	1.0065
EMWM12III	0.57265e-1	0.10604e-9	0.00000e+0	11.117
LMWM14I	0.41895e-3	0.00000e+0	0.00000e+0	14.001
LMWM14II	0.32328e-2	0.00000e+0	0.00000e+0	13.993
NMWM14	0.41987e-2	0.00000e+0	0.00000e+0	13.998

Table 2. Numerical results for $f_2(x)$.

Method	$\|x_1 - \xi\|$	$\|x_2 - \xi\|$	$\|x_3 - \xi\|$	COC
	$x_0 = 0.1,$	$\gamma_0 = 0.01,$	$\alpha_0 = 0.01,$	$\xi = 0$
JMWM6	0.58358e-2	0.13033e-9	0.72402e-11	0.2118
LMWM6	0.58908e-2	0.27816e-5	0.25682e-11	1.8159
CMWM7	0.57743e-2	0.72552e-10	0.48367e-11	0.1944
NMWM7	0.58035e-2	0.50597e-10	0.24093e-11	0.2093
LTMWM12I	0.47739e-3	0.10092e-8	0.70720e-19	1.7894
LTMWM12II	0.49877e-3	0.10930e-8	0.82960e-19	1.7882
LTMWM12III	0.49306e-3	0.10704e-8	0.79560e-19	1.7886
LTMWM12IV	0.48311e-3	0.10313e-8	0.73864e-19	1.7891
EMWM12I	0.26019e-3	0.28620e-9	0.54599e-20	1.7990
EMWM12II	0.26008e-3	0.28586e-9	0.54466e-20	1.7990
EMWM12III	0.26019e-3	0.28620e-9	0.54599e-20	1.7990
LMWM14I	0.46693e-3	0.10356e-35	0.71493e-73	1.1380
LMWM14II	0.44276e-3	0.26703e-36	0.47536e-74	1.1364
NMWM14	0.29387e-2	0.15456e-8	0.14567e-87	11.966

4 Summary

In this study, convergence order of the existing fourth-and eighth-order derivative free methods has been improved without any extra evaluation. As a result the efficiency is also increased. To justify the theoretical convergence order two non-smooth functions are presented. The numerical results show that proposed method is very useful to find an acceptable approximation of the exact solution of nonlinear equations, specially when the function is non-differentiable.

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