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Oscillation of a Nonlinear Neutral Difference Equation with Positive and Negative Coefficients

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Abstract This paper is concerned with the oscillatory behavior of a class of neutral nonlinear difference equation with positive and negative coefficients. The oscillation criteria are given, which extended the results of known ones.

Keywords: Neutral, nonlinear difference equation, positive and negative coefficient

1 Introduction

Oscillatory behavior is an important topic for difference equations, which has attracted many researchers for many years, see for example [1,2,3,4,5,6] and references therein. In [5], the author discussed the oscillation property for a neutral linear difference equation with positive and negative coefficients

$$\Delta[x(n) - r(n)x(n-\xi)] + p(n)x(n-\tau) - q(n)x(n-\sigma) = 0, n = 0, 1, 2, ..., \tag{1}$$

where ξ, τ, σ are positive integers and that $\tau \geq \sigma$, $r(n)$ is a real sequence, $p(n)$ and $q(n)$ are nonnegative sequences. Note that (1) is linear. Motivated by [5], in this paper, we consider the oscillation of a class of nonlinear neutral difference equation with positive and negative coefficients

$$\Delta[x(n) - R(n)x(n-r)] + P(n)f(x(n-k)) - Q(n)f(x(n-l)) = 0, n \geqslant n_0 \tag{2}$$

where $R(n), P(n)$ and $Q(n)$ are nonnegative sequences, r, k, l, n_0 are positive integers with $k \geq l+1$, and $f \in C(R, R)$. To our knowledge, no any oscillation results for nonlinear equation (1) with positive and negative coefficients has been presented.

2 Main Results

We make the following assumptions:

(H1) $P(n), Q(n)$ and $R(n)$ are nonnegative sequences, $P(n)$ and $R(n) + Q(n)$ have positive subsequences;
(H2) $H(n) := P(n+k-l) - Q(n)$ is nonnegative and has a positive subsequence;
(H3) There exists a constant $A > 0$ such that $|f(x)| \leq A|x| (x \in R)$, $xf(x) > 0$ for $x \neq 0, x \in \mathbf{R}$;

Let

$$m_0 := \begin{cases} k, & R \equiv 0, \\ \max\{r, k\}, & \text{others}; \end{cases} \qquad m_1 := \begin{cases} l, & R \equiv 0, \\ r, & Q \equiv 0, \\ \min\{r, l\}, & \text{others}; \end{cases}$$

and

$$z_x(n) := x(n) - R(n)x(n-r) - \sum_{i=n-k+l}^{n-1} P(i+k-l)f(x(i-l)), n \geqslant m_0 + n_0. \tag{3}$$

.

Consider the following inequality

$$\Delta[x(n) - R(n)x(n-r)] + P(n)f(x(n-k)) - Q(n)f(x(n-l)) \leqslant 0, n \geqslant n_0. \tag{4}$$

We have

Lemma 1 Assume that $\{x(n)\}$ is an eventual positive solution of (4). If

$$R(n)+A\sum_{i=n-k+l}^{n-1}Q(i)\leq 1 \tag{5}$$

holds for n large enough, then $\Delta z_x(n)\leq 0$, $z_x(n)>0$ for n large enough.

Proof. Let $\{x(n)\}$ be a solution of the difference inequality (4). Then there exists $n_1\geq n_0$ such that $x(n-m_0)>0$ for $n\geq n_1$. By(4) and(3), we have

$$\begin{aligned}\Delta z_x(n)&=\Delta[x(n)-R(n)x(n-r)-\sum_{i=n-k+l}^{n-1}P(i+k-l)f(x(i-l))]\\&=\Delta[x(n)-R(n)x(n-r)]-P(n+k-l)f(x(n-l))+P(n)f(x(n-k))\\&\leq -P(n+k-l)f(x(n-l))+Q(n)f(x(n-l))\\&\leq -H(n)f(x(n-l))\leq 0.\end{aligned} \tag{6}$$

So either $z_x(n)$ is eventually negative or $z_x(n)$ is eventually positive.

We first assume $z_x(n)$ is eventually negative. Then there exists $n_2\geq n_1$, $\alpha>0$ such that $z_x(n_2)\leq -\alpha<0$ for $n\geq n_2$. Summing of two sides of (6) from $i=n_2$ to $n-1$ yields

$$z_x(n)\leq z_x(n_2)-\sum_{i=n_2}^{n-1}H(i)f(x(i-l))\leq -\alpha-\sum_{i=n_2}^{n-1}H(i)f(x(i-l)).$$

Thus for $n>n_3\geq n_2+m_0$, we have

$$\begin{aligned}x(n)&\leq -\alpha+R(n)x(n-r)+\sum_{i=n-k+l}^{n-1}P(i+k-l)f(x(i-l))-\sum_{i=n_2}^{n-1}H(i)f(x(i-l))\\&\leq -\alpha+R(n)x(n-r)+\sum_{i=n-k+l}^{n-1}Q(i)f(x(i-l)).\end{aligned} \tag{7}$$

Accordingly, we consider the following two cases.

Case 1. $\{x(n)\}$ is unbounded. Then $\limsup_{n\to\infty}x(n)=\infty$. Hence, there exists a subsequence $\{s_j\}_{j=1}^{\infty}$ such that $x(s_j)\to\infty$ as $s_j\to\infty$. Take $s_j\geq n_3+m_0$. Let $x(t_j)=\max\{x(n):n_2\leq n\leq s_j\}$, $j=1,2,3,\cdots$.

By (5) and (7), we have

$$x(t_j)\leq -\alpha+R(s_j)x(t_j-r)+\sum_{i=t_j-k+l}^{t_j-1}Q(i)f(x(i-l))\leq -\alpha+x(t_j),$$

which is a contradiction.

Case 2. $\{x(n)\}$ is bounded. Then $\limsup_{n\to\infty}x(n)=a<\infty$, and there exists a subsequence $\{\overline{s}_j\}_{j=1}^{\infty}$ such that $\overline{s}_j\to\infty, x_{\overline{s}_j}\to a$ as $j\to\infty$. Let ξ_j be the ones such that $x(\xi_j)=\max\{x_s:\overline{s}_j-m_0\leq s\leq\overline{s}_j-m_1\}$, $j=1,2,3,\cdots$. Thus $\xi_j\to\infty$ and $\limsup_{j\to\infty}x(\xi_j)\leq a$ as $j\to\infty$.

By (5) and (7), we have

$$x(\overline{s}_j)\leq -\alpha+R(\overline{s}_j)x(\overline{s}_j-r)+\sum_{i=\overline{s}_j-k+l}^{\overline{s}_j-1}Q(i)f(x(i-l))\leq -\alpha+x(\xi_j),$$

and further,

$$a\leq -\alpha+\limsup_{j\to\infty}x(\xi_j)\leq -\alpha+a,$$

which is also a contradiction.

The above discussions finish the proof. □

Lemma 2. Assume that $\{x(n)\}$ is an eventually positive solution of the difference inequality (4). If

$$R(n) + A \sum_{i=n-k+l}^{n-1} P(i+k-l) \geq 1 \tag{8}$$

holds for n big enough, then either

$$\liminf_{n\to\infty} x(n) > 0 \tag{9}$$

or $z_x(n) < 0$ holds for n big enough.

Proof. Since $\{x(n)\}$ an eventually positive solution of the difference inequality (4), then there exists $n_1 \geq n_0$ such that $x(n-m_0) > 0$ for $n \geq n_1$. It is easy to know (6) holds. So there exits $n_2 \geq n_1$ such that $z_x(n) < 0$ or $z_x(n) > 0$ is true for $n \geq n_2$.

Without loss the generality, we assume $z_x(n) > 0$. Let $M := \frac{1}{2}\min\{x(i) : n_2 - m_0 \leq i \leq n_2\}$ Then $M \geq \frac{1}{2} z_x(n_2) > 0$, and

$$x(n) > \frac{1}{A} f(x(n)) > M. \tag{10}$$

for $n \geq n_2$. Indeed, if there exists $n_3 > n_2$ such that $x(n) > \frac{1}{A} f(x(n))$ with $x(n) \leq M$ for $n \in [n_2, n_3)$. By (3) and (8), we have

$$\begin{aligned} M \geq x(n_3) &= z_x(n_3) + R(n_3)x(n_3 - r) + \sum_{i=n_3-k+l}^{n_3-1} P(i+k-l) f(x(i-l)) \\ &> R(n_3)x(n_3 - r) + \sum_{i=n_3-k+l}^{n_3-1} P(i+k-l) f(x(i-l)) \\ &> [R(n_3) + A \sum_{i=n_3-k+l}^{n_3-1} P(i+k-l)]M \geq M, \end{aligned}$$

which is a contradiction. So $\liminf_{n\to\infty} x(n) > 0$. □

By Lemma 1 and Lemma 2, we can obtain the following

Lemma 3. Assume that $\{x(n)\}$ is an eventually positive solution of the difference inequality (4). If (5) and (8) hold for n big enough, then

$$\liminf_{n\to\infty} x(n) > 0 \quad \text{and} \quad \Delta z_x(n) \leq 0, \quad z_x(n) > 0. \tag{11}$$

Similar to [5], we can easily obtain

Lemma 4. Assume conditions $H(1) - H(3)$ are satisfied for n big enough, and (5) holds. Then any solution of (1) is oscillatory if and only if the difference inequity (4) has no eventually positive solution.

Theorem 1. Assume (5) and (8) hold eventually and

$$\sum_{i}^{\infty} H(i) = \infty. \tag{12}$$

Then all solutions of (1) are oscillatory.

Proof. We shall prove by the contradiction. Let $\{x(n)\}$ is an eventually positive solution of the difference inequality (4). Assume $x(n-m_0) > 0$ for $n_1 \geq n_0, n \geq n_1$. By (12), summing two sides of (6) from $i = n_1$ to ∞ leads to

$$z_x(n_1) \geq \sum_{i=n_1}^{\infty} H(i) f(x(i-l)) = \infty,$$

which is contradictory to Lemma 3. The proof is complete. □

Example 1. Consider the following difference equation

$$\Delta[x(n) - r(n)x(n-r)] + p(n)\sin(x(n-k)) - q(n)\sin(x(n-l)) = 0, n \geqslant n_0, \tag{13}$$

where $p(n) = 2n$, $q(n) = \frac{1}{n}$, $r(n) = \frac{(-1)^n+1}{n}$ and $H(n) = \frac{2(n+1)^2-3}{n}$. Taking $k - l = 2$, then all conditions of Theorem 1 hold. So all solutions of (1) are oscillatory.

3 Conclusion

Based on the oscillatory property of the difference inequalities, this paper considered the oscillation behavior of a nonlinear neutral difference equation with positive and negative coefficients. Under some conditions, the difference equation discussed is oscillatory. Necessary oscillatory conditions for (1) will be considered later.

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