L_p-Convergence of Orthogonal Polynomial Expansions for Exponential Weights

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Abstract Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in \mathbf{C^1}(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function which is an exponent. We consider the weight $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$. Let us denote the partial sum of Fourier series for a function f by $s_n(f;x) := s_n(f;w^2;x)$, and the de la Vallée Poussin mean of f by $v_n(f) := v_n(f;w^2)$. Then we investigate the convergences of $s_n(f)$ and $v_n(f)$ with $w(x)$.

Keywords: orthogonal polynomial expansions

1 Introduction

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in \mathbf{C^1}(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function. We consider the weight $w(x)$;

$$w(x) := \exp(-Q(x)), \quad x \in \mathbb{R}.$$

Then we suppose that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$.

Now we can construct the orthonormal polynomials $p_n(x) = p_n(w^2; x)$ of degree n for $w^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w^2(x)dx = \delta_{mn} \quad \text{(Kronecker delta)}.$$

For the weight w we define the partial sum of Fourier series of f by

$$s_n(f)(x) := \sum_{k=0}^{n-1} b_k(f)p_k(x), \quad b_k(f) = \int_{\mathbb{R}} f(t)p_k(t)w^2(t)dt$$

for $n \in \mathbb{N}$. Then we also the de la Vallée Poussin mean $v_n(f)$ of f is defined by

$$v_n(f)(x) := \frac{1}{n} \sum_{j=n+1}^{2n} s_j(f)(x).$$

We say that $f : \mathbb{R} \to [0, \infty)$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leqslant Cf(y)$ for $0 < x < y$.

First we need the following definition from [5].

Definition 1.1. The weight $w(x) = \exp(-Q(x))$ satisfies the following. Let $Q : \mathbb{R} \to [0, \infty)$ be a continuous and an even function, and satisfy the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R}\backslash\{0\}$.

(c)

$$\lim_{x\to\infty} Q(x) = \infty.$$

(d) The function

$$T(x) := T_w(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{1.1}$$

is quasi-increasing in $(0, \infty)$, with

$$T(x) \geqslant \Lambda > 1, \quad x \in \mathbb{R}\backslash\{0\}. \tag{1.2}$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leqslant C_1 \frac{|Q'(x)|}{Q(x)}, \quad a.e.\ x \in \mathbb{R}\backslash\{0\}. \tag{1.3}$$

Then we write $w = \exp(-Q) \in \mathcal{F}(C^2)$. If there also exists a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geqslant C_2 \frac{|Q'(x)|}{Q(x)}, \quad a.e.\ x \in \mathbb{R}\backslash J, \tag{1.4}$$

then we write $w = \exp(-Q) \in \mathcal{F}(C^2+)$.
Example 1.2. (1) If an exponential $Q(x)$ satisfies

$$1 < \Lambda_1 \leqslant \frac{(xQ'(x))'}{Q'(x)} \leqslant \Lambda_2,$$

where $\Lambda_i,\ i = 1, 2$ are constants, then we call $w = \exp(-Q(x))$ the Freud weight. The class $\mathcal{F}(C^2+)$ contains the Freud-type weights.
(2) For $\alpha > 1,\ r \geqslant 1$ we define

$$Q(x) = Q_{r,\alpha}(x) = \exp_r(|x|^\alpha) - \exp_r(0),$$

where $\exp_r(x) = \exp(\exp(\exp \ldots \exp x) \ldots)$ (r times). Moreover, we define

$$Q_{r,\alpha,m}(x) = |x|^m\{\exp_r(|x|^\alpha) - \alpha^* \exp_r(0)\}, \quad \alpha + m > 1,\ m \geqslant 0,\ \alpha \geqslant 0,$$

where $\alpha^* = 0$ if $\alpha = 0$, and otherwise $\alpha^* = 1$.
(3) We define

$$Q_\alpha(x) = (1 + |x|)^{|x|^\alpha} - 1, \quad \alpha > 1.$$

If $T(x)$ is bounded, then we call w the Freud-type weight, and if $T(x)$ is unbounded, then we call w the Erdös-type weight.
Definition 1.3. Let $w = \exp(-Q) \in \mathcal{F}(C^2+), 1 < \lambda < \frac{m+2}{m+1}$ and $m \geqslant 1$ be an integer. Then we write $w \in \mathcal{F}_\lambda(C^{m+2}+)$ if $Q \in C^{m+2}(\mathbb{R})$ and there exist constants $C \geqslant 1$ and $K \geqslant 1$ such that for all $|x| \geqslant K$

$$\frac{|Q'(x)|}{Q(x)^\lambda} \leqslant C \quad \text{and} \quad |\frac{Q''(x)}{Q'(x)}| \sim |\frac{Q^{(k+1)}(x)}{Q^{(k)}(x)}| \tag{1.5}$$

for every $k = 1, ..., m$ and also

$$|\frac{Q^{(m+2)}(x)}{Q^{(m+1)}(x)}| \leqslant \frac{Q^{(m+1)}(x)}{Q^{(m)}(x)}|. \tag{1.6}$$

In particular, $w \in \mathcal{F}_\lambda(C^3+)$ means that $Q \in C^3(\mathbb{R})$ and

$$\frac{|Q'(x)|}{Q(x)^\lambda} \leqslant C\ (1 < \lambda < 3/2) \quad \text{and} \quad |\frac{Q'''(x)}{Q''(x)}| \leqslant C|\frac{Q''(x)}{Q'(x)}| \tag{1.7}$$

hold for $|x| \geq K > 0$.
In [3] we obtain the following result.
Theorem 1. ([3, Theorem 1.1]) Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$. Suppose that f is continuous and has a bounded variation on any compact interval of $\mathbb{R}$. If f satisfies

$$\int_{-\infty}^{\infty} w(x)|df(x)| < \infty,$$

then

$$\lim_{n\to\infty}\|(f-s_n(f))\frac{w}{T^{1/4}}\|_{L_\infty(\mathbb{R})}=0.$$

First, we extend Theorem 1 to L_p−space. To do so we need to define a new weight class.

Definition 1.4. For a weight $w=\exp(-Q)$, we set

$$\lambda_w:=\lim\sup_{x\to\infty}\frac{Q''(x)Q(x)}{Q'(x)^2}\quad\text{and}\quad\mu_w:=\lim\inf_{x\to\infty}\frac{Q''(x)Q(x)}{Q'(x)^2}.$$

If $\lambda_w=\mu_w$ holds, then we say that a weight w is regular.

All the weights in Example 1.2 are regular.

The Mhaskar-Rakhmanov-Saff number (MRS number) a_t is defined by

$$t=\frac{2}{\pi}\int_0^1\frac{a_tuQ'(a_tu)}{(1-u^2)^{1/2}}du,t>0.$$

Lemma 1.5. ([9, Corollary 5.5]) Let w be a regular weight. Then for any $\delta>0$ there exists a constant $C>0$ such that

$$T(a_t)\leqslant Ct^\delta,\quad t\geqslant C.$$

Now, we can extend Theorem 1 to L_p-space.

Theorem 2. Let $w=\exp(-Q)$ be a regular weight and $w\in\mathcal{F}_\lambda(C^3+)$ with $1<\lambda<3/2$. Suppose that f is continuous and has a bounded variation on any compact interval of $\mathbb{R}$. Let $1\leqslant p<\infty$. We suppose that f satisfies

$$\int_{-\infty}^{\infty}w(x)|df(x)|<\infty.$$

If w is an Erdös-type weight, then we have

$$\lim_{n\to\infty}\|(f-s_n(f))\frac{w}{T^{1/4}}\|_{L_p(\mathbb{R})}=0,\tag{1.8}$$

and if w is a Freud-type weight, then for $2/\Lambda\leqslant p<\infty$, where Λ is defined by Definition 1.1 (d), we have (1.8).

For $f\in C(\mathbb{R})$, the degree of weighted polynomial approximation is defined by

$$E_{p,n}(w;f):=\inf_{P\in\mathcal{P}_n}\|w(f-P)\|_{L_p(\mathbb{R})},\quad\text{where}\quad 1\leqslant p\leqslant\infty.$$

Especially, if $p=\infty$, then we write $E_n(w;f):=E_{\infty,n}(w;f)$.

With respect to $v_n(f)$, we have the following convergence theorem.

Theorem 3. We suppose $w\in\mathcal{F}_\lambda(C^4+)$ with $1<\lambda<4/3$, furthermore we assume $T(a_n)\leqslant C(\frac{n}{a_n})^{2/3}$. Let $\nu\geqslant 0$ be an integer, and let $1\leqslant p\leqslant\infty$. We suppose that $f\in C^\nu(\mathbb{R})$ with $\|T^{(2\nu+1)/4}f^{(\nu)}w\|_{L_\infty(\mathbb{R})}<\infty$. Then we have for $\beta>1$ and $0\leqslant j\leqslant\nu$,

$$\begin{aligned}&\|(f^{(j)}(x)-v_n^{(j)}(f;x))w(x)(1+|x|)^{\beta/p}\|_{L_p(\mathbb{R})}\\&\leqslant C_\nu(\frac{a_n}{n})^{\nu-j}T(a_n)^{*1/4}E_{n-\nu}(T^{(2\nu+1)/4}w;f^{(\nu)}),\end{aligned}\tag{1.9}$$

where

$$T(a_n)^{*1/4}=\begin{cases}1, & 0\leqslant j\leqslant\nu-1;\\ T(a_n)^{1/4}, & j=\nu.\end{cases}$$

Remark 1.6. Let $1\leqslant p\leqslant\infty$.

(1) For $0 \leqslant j \leqslant \nu - 1$, (1.9) means

$$\lim_{n\to\infty} \|(f^{(j)}(x) - v_n^{(j)}(f;x))w(x)(1+|x|)^{\beta/p}\|_{L_p(\mathbb{R})} = 0. \tag{1.10}$$

(2) We consider (1.9) for $j = \nu$. We suppose that $T^{(2\nu+1)/4}f^{(\nu)}w$ is continuous and

$$\lim_{|x|\to\infty} T^{(2\nu+1)/4}(x)f^{(\nu)}(x)w(x) = 0.$$

If w is a Freud-type weight, then we also have (1.10) for $j = \nu$. If w is an Erdös-type weight, then we further suppose that w is a regular weight and

$$E_{n-\nu}(T^{(2\nu-1)/4}w; f^{(\nu)}) \leqslant Cn^{-\beta}$$

for some $\beta > 0$. Under these conditions we have (1.10) with $j = \nu$.

Throughout this paper, $c, C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t or polynomials $P_n(x)$.

2 Proof of Theorem 2

In this section we prove Theorem 2. To prove the theorem we need some lemmas.

Lemma 2.1. ([2, Corollary 14]) We obtained the following result: Let $1 \leqslant p \leqslant \infty$. We assume that $w \in \mathcal{F}(C^2+)$ satisfies

$$T(a_n) \leqslant C(\frac{n}{a_n})^{2/3}. \tag{2.1}$$

Then there exists a constant $C = C(w,p) > 0$ such that, for every $n \in \mathbb{N}$ and every $wf \in L_p(\mathbb{R})$,

$$\|(f - v_n(f))\frac{w}{T^{1/4}}\|_{L_p(\mathbb{R})} \leqslant CE_{p,n}(w;f), \tag{2.2}$$

and when $T^{1/4}wf \in L_p(\mathbb{R})$, we have

$$\|(f - v_n(f))w\|_{L_p(\mathbb{R})} \leqslant CE_{p,n}(T^{1/4}w;f). \tag{2.3}$$

Remark 2.2. Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$, then (2.1) holds true (see [2, Remark 16]).

Lemma 2.3. Let $\Lambda > 1$ be defined in Definition 1.1 (d) and $a_n > 1$. Then we have

$$a_n \leqslant Cn^{1/\Lambda}. \tag{2.4}$$

Especially, if w is an Erdös-type weight, then for any $\eta > 0$ there exists $C_\eta > 0$ depending only on η such that

$$a_n \leqslant C_\eta n^\eta \tag{2.5}$$

(see [9, Lemma 3.2 (3.6)]).

Proof of (2.4). Let $x \geqslant 1$. From (d) in Definition 1.1, we have

$$\int_1^x \frac{Q'(t)}{Q(t)}dt \geqslant \Lambda \int_1^x \frac{1}{t}dt.$$

Hence we see

$$\frac{Q(x)}{Q(1)} \geqslant x^\Lambda,$$

so for $a_n > 1$ we have

$$\frac{Q(a_n)}{Q(1)} \geqslant a_n^\Lambda.$$

By [5, Lemma 3.4 (3.18)] we see

$$n \geqslant c\frac{n}{Q(1)\sqrt{T(a_n)}} \geqslant ca_n^{\Lambda}.$$

Therefore, we have (2.4). #

Lemma 2.4. ([7, Lemma 3.6]) Let $w \in \mathcal{F}(C^2+)$, $P \in \mathcal{P}_n$, and let $1 \leqslant p, q \leqslant \infty$. Then for $q \leqslant p$,

$$\|wP\|_{L_q(\mathbb{R})} \leqslant Ca_n^{\frac{1}{q}-\frac{1}{p}}\|wP\|_{L_p(\mathbb{R})}, \tag{2.6}$$

and for $p < q$,

$$\|\frac{w}{\sqrt{T}}P\|_{L_q(\mathbb{R})} \leqslant C(\frac{n}{a_n})^{\frac{1}{p}-\frac{1}{q}}\|wP\|_{L_p(\mathbb{R})}. \tag{2.7}$$

Lemma 2.5. ([3, Proof of Theorem 1.1 (5.3)])

$$\|(v_n(f) - s_n(f))w\|_{L_2(\mathbb{R})} = o(1)\sqrt{\frac{a_n}{n}}.$$

Lemma 2.6. ([9, Theorem 4.1 and (4.11)]) Let $1 < \lambda < 3/2$ and $\alpha \in \mathbb{R}$. Then for $w = \exp(-Q) \in \mathcal{F}(C^3+)$ we can construct a new weight $w_\alpha = \exp(-Q) \in \mathcal{F}(C^2+)$ such that

$$T(x)^\alpha w(x) \sim w_\alpha(x)$$

on $\mathbb{R}$ and

$$a_{n/C_0} \leqslant a_n(w_\alpha) \leqslant a_{C_0 n}.$$

Lemma 2.7. ([5, Theorem 1.9 (a)]) $w \in \mathcal{F}(C^2+)$, $0 < p \leqslant \infty$ and $P \in \mathcal{P}_n$ $(n \geqslant 1)$. Then

$$\|Pw\|_{L_p(\mathbb{R})} \leqslant 2\|Pw\|_{L_p(|x|\leqslant a_n)}.$$

Proof of Theorem 2. Let w be an Erdös-type weight. We see

$$\begin{aligned}
&\|(f - s_n(f))\frac{w}{T^{1/4}}\|_{L_p(\mathbb{R})} \\
&\leqslant \|(f - v_n(f))\frac{w}{T^{1/4}}\|_{L_p(\mathbb{R})} + \|(v_n(f) - s_n(f))\frac{w}{T^{1/4}}\|_{L_p(\mathbb{R})} \\
&\leqslant \|(f - v_n(f))\frac{w}{T^{1/4}}\|_{L_p(\mathbb{R})} + 2\|(v_n(f) - s_n(f))\frac{w}{T^{1/4}}\|_{L_p(|x|\leqslant a_{2n})}
\end{aligned}$$

by Lemma 2.7 with $\frac{w}{T^{1/4}} \sim w_{-1/4} \in \mathcal{F}(C^2+)$

$$\leqslant CE_{p,n}(w; f) + CT(a_n)^{1/4}\|(v_n(f) - s_n(f))\frac{w}{T^{1/2}}\|_{L_p(\mathbb{R})}$$

by Lemma 2.1 (2.2) (we note that (2.1) holds)

$$= o(1) + CT(a_n)^{1/4}\|(v_n(f) - s_n(f))\frac{w}{T^{1/2}}\|_{L_p(\mathbb{R})} \tag{2.8}$$

(see [6, Theorem 1.4 and 1.6] about $E_{p,n}(w; f) \to 0$ as $n \to \infty$). From Lemma 2.4 (2.7) and Lemma 2.5 we see

$$\begin{aligned}
&T(a_n)^{1/4}\|(v_n(f) - s_n(f))\frac{w}{T^{1/2}}\|_{L_p(\mathbb{R})} \leqslant CT(a_n)^{1/4}(\frac{n}{a_n})^{\frac{1}{2}-\frac{1}{p}}\|(v_n(f) - s_n(f))w\|_{L_2(\mathbb{R})} \\
&\leqslant CT(a_n)^{1/4}(\frac{n}{a_n})^{\frac{1}{2}-\frac{1}{p}}o(1)\sqrt{\frac{a_n}{n}} = o(1)T(a_n)^{1/4}(\frac{a_n}{n})^{1/p} = o(1)
\end{aligned}$$

by Lemma 1.5. Therefore, (2.8) means

$$\lim_{n\to\infty}\|(f - s_n(f))\frac{w}{T^{1/4}}\|_{L_p(\mathbb{R})} = 0. \tag{2.9}$$

Let w be a Freud-type weight. If $2 < p < \infty$, then as above we have (2.9) because of $T(x) \sim 1$. Let $2/\Lambda \leqslant p \leqslant 2$. Then by Lemma 2.4 (2.6), Lemma 2.5 and Lemma 2.3 (2.4) we see

$$\|(v_n(f) - s_n(f))\frac{w}{T^{1/2}}\|_{L_p(\mathbb{R})} \leqslant Ca_n^{\frac{1}{p}-\frac{1}{2}}\|(v_n(f) - s_n(f))w\|_{L_2(\mathbb{R})}$$
$$= o(1)a_n^{\frac{1}{p}-\frac{1}{2}}\sqrt{\frac{a_n}{n}} = o(1)a_n^{1/p}\sqrt{\frac{1}{n}} = o(1)n^{1/p\Lambda}\sqrt{\frac{1}{n}} = o(1) \tag{2.10}$$

because of $p\Lambda \geqslant 2$. Consequently, from (2.8)-(2.10) we have the result. #

3 Proof of Theorem 3

To prove the theorem we need some lemmas.
Lemma 3.1. ([1, Theorem 1.2]) Let $\nu \geqslant 0$. We suppose that $w \in \mathcal{F}_\lambda(C^4+),\ 1 < \lambda < (\nu+4)/(\nu+3)$. Let $\|T^{(2\nu+1)/4}fw\|_{L_\infty(\mathbb{R})} < \infty$ with an integer $\nu \geqslant 0$. Then there is a constant $C > 1$ such that for $0 \leqslant j \leqslant \nu$,

$$\|v_n^{(j)}(f)w\|_{L_\infty(\mathbf{R})} \leqslant C(\frac{n}{a_n})^j\|T^{(2j+1)/4}fw\|_{L_\infty(\mathbf{R})} \tag{3.1}$$

holds for all $n \in \mathbb{N}$.
Lemma 3.2. (cf. [4, Theorem 2.3]) Let $\nu \geqslant 0$ be an integer. Let

$$w = \exp(-Q) \in \mathcal{F}_\lambda(C^4+),\ 1 < \lambda < 4/3. \tag{3.2}$$

Suppose that $f \in C^\nu(\mathbb{R})$ with

$$\|T^{1/4}f^{(\nu)}w\|_{L_\infty(\mathbb{R})} < \infty.$$

Then there exists an absolute constant $C_\nu > 0$ such that for $0 \leqslant k \leqslant \nu$ and $x \in \mathbb{R}$,

$$|(f^{(k)}(x) - P_{n,f,w}^{(k)}(x))w(x)| \leqslant CT^{k/2}(x)E_{n-k}(w_{1/4}, f^{(k)}), \tag{3.3}$$

where $T^{1/4}(x)w(x) \sim w_{1/4} \in \mathcal{F}(C^2+)$.
Proof. Let $\nu \geqslant 0$. [4, Theorem 2.3] states that under the condition;

$$w \in \mathcal{F}_\lambda(C^{\nu+3}+),\ 1 < \lambda < (\nu+3)/(\nu+2), \tag{3.4}$$

(3.3) holds (see Appendix; Theorem C). In [Appendix; Theorem D] we will show that the assumption (3.4) is reduced to the assumption (3.2). #
Lemma 3.3. [8, Theorem 1 and Corollary 8] Let $w \in \mathcal{F}(C^2+)$. Let f be $s-1$ times continuously differentiable for some integer $s \geqslant 0$, and let $f^{(s-1)}(x)$ be absolutely continuous in each compact interval (Here we omit this condition if $s = 0$). Let $wf^{(s)} \in L_\infty(\mathbb{R})$. Then we have

$$E_n(w; f) \leqslant C(\frac{a_n}{n})^s\|wf^{(s)}\|_{L_\infty(\mathbb{R})},$$

equivalently,

$$E_n(w; f) \leqslant C(\frac{a_n}{n})^sE_{n-s}(w; f^{(s)}).$$

Proof of Theorem 3. Let $1 \leqslant p < \infty$ and $0 \leqslant j \leqslant \nu$. We easily see that by $\beta > 1$,

$$\|(f^{(j)}(x) - v_n^{(j)}(f; x))w(x)(1 + |x|)^{\beta/p}\|_{L_p(\mathbb{R})}$$
$$\leqslant \|(f^{(j)}(x) - v_n^{(j)}(f; x))w(x)\|_{L_\infty(\mathbb{R})}\|(1 + |x|)^{\beta/p}\|_{L_p(\mathbb{R})}$$
$$\leqslant C\|(f^{(j)}(x) - v_n^{(j)}(f; x))w(x)\|_{L_\infty(\mathbb{R})}.$$

Therefore, for $T^{(2\nu+1)/4}w \sim w_{(2\nu+1)/4} \in \mathcal{F}(C^2+)$ we may show

$$\|(f^{(j)} - v_n^{(j)}(f))w\|_{L_\infty(\mathbb{R})} \leqslant C_\nu(\frac{a_n}{n})^{\nu-j}T(a_n)^{*1/4}E_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)}). \tag{3.5}$$

Let $P_{n,f,w_{\nu/2}} \in \mathcal{P}_n$ be the best approximation polynomial for f with respect to the weight $w_{\nu/2}$. First, we rewrite (3.3) as follows. Using $w_{\nu/2}$, we have

$$|(f^{(j)}(x) - P_{n,f,w_{\nu/2}}^{(j)}(x))w_{\nu/2}(x)| \leqslant CT^{j/2}(x)E_{n-j}(w_{(2\nu+1)/4}; f^{(j)}). \tag{3.6}$$

Hence, we see

$$\begin{aligned}
&|(f^{(j)}(x) - P_{n,f,w_{\nu/2}}^{(j)}(x))w(x)| \leqslant |(f^{(j)}(x) - P_{f,w_{\nu/2}}^{(j)}(x))w_{(\nu-j)/2}(x)| \\
&\leqslant CT(x)^{-j/2}|(f^{(j)}(x) - P_{n,f,w_{\nu/2}}^{(j)}(x))w_{\nu/2}(x)| \\
&\leqslant CE_{n-j}(w_{(2\nu+1)/4}; f^{(j)}).
\end{aligned} \tag{3.7}$$

For $j \leqslant \nu - 1$ we see

$$\begin{aligned}
&|(f^{(j)}(x) - v_n^{(j)}(f;x))w(x)| \\
&= |(f^{(j)}(x) - P_{n,f,w_{\nu/2}}^{(j)}(x) - v_n^{(j)}(f - P_{n,f,w_{\nu/2}}(x))w(x)| \\
&\leqslant |(f^{(j)}(x) - P_{n,f,w_{\nu/2}}^{(j)})(x)w(x)| + \|v_n^{(j)}(f - P_{n,f,w_{\nu/2}})w\|_{L_\infty(\mathbb{R})} \\
&\leqslant CE_{n-j}(w_{(2\nu+1)/4}; f^{(j)}) + C(\frac{n}{a_n})^j\|(f - P_{n,f,w_{\nu/2}})w_{(2j+1)/4}\|_{L_\infty(\mathbb{R})}
\end{aligned}$$

by (3.7) and (3.1)

$$\begin{aligned}
&\leqslant CE_{n-j}(w_{(2\nu+1)/4}; f^{(j)}) + C(\frac{n}{a_n})^j\|(f - P_{n,f,w_{\nu/2}})w_{\nu/2}\|_{L_\infty(\mathbb{R})} \\
&\leqslant CE_{n-j}(w_{(2\nu+1)/4}; f^{(j)}) + C(\frac{n}{a_n})^jE_n(w_{\nu/2}; f) \\
&\leqslant CE_{n-j}(w_{(2\nu+1)/4}; f^{(j)}) + CE_{n-j}(w_{\nu/2}; f^{(j)}) \\
&\leqslant C(\frac{a_n}{n})^{\nu-j}E_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)})
\end{aligned}$$

by Lemma 3.3. Let $j = \nu$. As above we have

$$\begin{aligned}
&|(f^{(\nu)}(x) - v_n^{(\nu)}(f;x))w(x)| \\
&= |(f^{(\nu)}(x) - P_{n,f,w_{\nu/2}}^{(\nu)}(x) - v_n^{(\nu)}(f - P_{n,f,w_{\nu/2}};x))w(x)| \\
&\leqslant T(x)^{-\nu/2}|(f^{(\nu)}(x) - P_{n,f,w_{\nu/2}}^{(\nu)}(x))w_{\nu/2}(x)| \\
&\quad + \|v_n^{(\nu)}(f - P_{n,f,w_{\nu/2}})w\|_{L_\infty(\mathbf{R})} \\
&\leqslant CE_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)}) + C(\frac{n}{a_n})^\nu\|(f - P_{n,f,w_{\nu/2}})w_{(2\nu+1)/4}\|_{L_\infty(\mathbb{R})}
\end{aligned}$$

by (3.6) with $j = \nu$ and (3.1)

$$\begin{aligned}
&\leqslant CE_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)}) \\
&\quad + C(\frac{n}{a_n})^\nu\|\{f - P_{n,f,w_{(2\nu+1)/4}} + (P_{f,w_{(2\nu+1)/4}} - P_{f,w_{\nu/2}})\}w_{(2\nu+1)/4}\|_{L_\infty(\mathbb{R})} \\
&\leqslant CE_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)}) + (\frac{n}{a_n})^\nu\|(f - P_{n,f,w_{(2\nu+1)/4}})w_{(2\nu+1)/4}\|_{L_\infty(\mathbb{R})} \\
&\quad + C(\frac{n}{a_n})^\nu\|(P_{n,f,w_{(2\nu+1)/4}} - P_{n,f,w_{\nu/2}})w_{(2\nu+1)/4}\|_{L_\infty(|x|\leqslant a_n)}
\end{aligned}$$

by Lemma 2.7

$$\leqslant CE_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)}) + E_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)})]$$
$$+C(\frac{n}{a_n})^{\nu}T(a_n)^{1/4}\|(P_{n,f,w_{(2\nu+1)/4}} - P_{n,f,w_{\nu/2}})w_{\nu/2}\|_{L_\infty(|x|\leqslant a_n)}$$

by Lemma 3.3

$$= CE_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)})$$
$$+C(\frac{n}{a_n})^{\nu}T(a_n)^{1/4}\|(-f + P_{n,f,w_{(2\nu+1)/4}} + f - P_{n,f,w_{\nu/2}})w_{\nu/2}\|_{L_\infty(\mathbb{R})}$$
$$\leqslant C[E_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)})$$
$$+(\frac{n}{a_n})^{\nu}T(a_n)^{1/4}\{E_n(w_{(2\nu+1)/4}; f) + E_n(w_{\nu/2}; f)\}]$$
$$\leqslant C_\nu T(a_n)^{1/4}E_{n-\nu}(w_{(2\nu+1)/4}; f^{(\nu)})]$$

by Lemma 3.3.

Consequently we have (3.5), that is, (1.9). #

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Appendix

In this appendix we prove Lemma 3.2 which is an improvement of [9, Theorem 4.2]. We need to prepare some results.

Theorem A. (cf. [9, Theorem 4.2]) Let

$$w = \exp(-Q) \in \mathcal{F}_\lambda(C^3+), \quad 1 < \lambda < 3/2, \tag{A.1}$$

and let w be an Erdös-type weight. Let $\mu,\ \nu,\ \alpha,\ \beta \in \mathbb{R}$. Then we can construct $w_{\mu,\nu,\alpha,\beta} \in \mathcal{F}(C^2+)$ such that

$$T_w(x)^\mu(1+x^2)^\nu(1+Q(x))^\alpha(1+|Q'(x)|)^\beta w(x) \sim w_{\mu,\nu,\alpha,\beta}(x) \tag{A.2}$$

on $\mathbb{R}$. And for some $0 < c \leqslant C$ we have

$$a_{cn}(w) \leqslant a_{w_{\mu,\nu,\alpha,\beta}}(w_{\mu,\nu,\alpha,\beta}) \leqslant a_{Cn}(w), \tag{A.3}$$

on $\mathbb{N}$, and

$$T_{w_{\mu,\nu,\alpha,\beta}}(x) \sim T_w(x) \tag{A.4}$$

on $\mathbb{R}$. Furthermore, if we suppose $w \in \mathcal{F}_\lambda(C^4+),\ 1 < \lambda < 4/3$, then

$$w_{\mu,\nu,\alpha,\beta} \in \mathcal{F}_\lambda(C^3+). \tag{A.5}$$

Proof. We suppose (A.1). To prove this theorem we apply the method of [9, Proof of Theorem 4.1]. We consider

$$q_{\mu,\nu,\alpha,\beta}(x) := \mu \log T(x) + \nu \log(1+x^2) + \alpha \log(1+Q(x)) + \beta \log(1+|Q'(x)|).$$

For $x \geqslant r$, where $r > 0$ large enough, we consider

$$\begin{aligned} &q_{\mu,\nu,\alpha,\beta}(x) \\ &= \{\mu \log x + \nu \log(1+x^2)\} + \{-\mu \log Q(x) + \alpha \log(1+Q(x))\} \\ &\quad + \{\mu \log Q'(x) + \beta \log(1+Q'(x))\} =: s(x) + u(x) + v(x). \end{aligned} \tag{A.6}$$

For this formula $q_{\mu,\nu,\alpha,\beta}$ we put the proof into practice as [9, Proof of Theorem 4.1]. We take a polynomial

$$p_{\mu,\nu,\alpha,\beta}(x) = p_{\mu,\nu,\alpha,\beta,r}(x) = (2r-x)^3(ax^2+bx+c)$$

such that

$$p_{\mu,\nu,\alpha,\beta}(r) = q_{\mu,\nu,\alpha,\beta}(r),\ {p'}_{\mu,\nu,\alpha,\beta}(r) = {q'}_{\mu,\nu,\alpha,\beta}(r),\ {p''}_{\mu,\nu,\alpha,\beta}(r) = {q''}_{\mu,\nu,\alpha,\beta}(r).$$

Now we set

$$Q_{\mu,\nu,\alpha,\beta}(x) := \begin{cases} Q(x), & \text{if } |x| \leqslant r; \\ Q(x) - q_{\mu,\nu,\alpha,\beta}(x) + p_{\mu,\nu,\alpha,\beta}(x), & \text{if } r < |x| \leqslant 2r; \\ Q(x) - q_{\mu,\nu,\alpha,\beta}(x), & \text{if } 2r < |x|. \end{cases}$$

We see

$$\begin{aligned} p_{\mu,\nu,\alpha,\beta}(x) = &\frac{1}{2r^5}(2r-x)^3[\{12q_{\mu,\nu,\alpha,\beta}(r) + 6{q'}_{\mu,\nu,\alpha,\beta}(r) + {q''}_{\mu,\nu,\alpha,\beta}(r)r^2\}x^2 \\ &-2\{9q_{\mu,\nu,\alpha,\beta}(r) + 5{q'}_{\mu,\nu,\alpha,\beta}(r) + {q''}_{\mu,\nu,\alpha,\beta}(r)r^2\}rx \\ &+\{8q_{\mu,\nu,\alpha,\beta}(r) + 4{q'}_{\mu,\nu,\alpha,\beta}(r) + {q''}_{\mu,\nu,\alpha,\beta}(r)r^2\}r^2], \end{aligned}$$

so that for every $x \in [r, 2r]$

$$|p_{\mu,\nu,\alpha,\beta}(x)| \leqslant C\{|q_{\mu,\nu,\alpha,\beta}(r)| + r|{q'}_{\mu,\nu,\alpha,\beta}(r)| + r^2|{q''}_{\mu,\nu,\alpha,\beta}(r)|\}, \tag{A.7}$$

where $C > 1$ is a constant independent of r. Similarly we have

$$|{p'}_{\mu,\nu,\alpha,\beta}(x)| \leqslant \frac{C}{r}\{|q_{\mu,\nu,\alpha,\beta}(r)| + r|{q'}_{\mu,\nu,\alpha,\beta}(r)| + r^2|{q''}_{\mu,\nu,\alpha,\beta}(r)|\} \tag{A.8}$$

and

$$|{p''}_{\mu,\nu,\alpha,\beta}(x)| \leqslant \frac{C}{r^2}\{|q_{\mu,\nu,\alpha,\beta}(r)| + r|{q'}_{\mu,\nu,\alpha,\beta}(r)| + r^2|{q''}_{\mu,\nu,\alpha,\beta}(r)|\}. \tag{A.9}$$

We shall show that if we take $r = r_{\mu,\nu,\alpha,\beta} > 0$ large enough, then $Q_{\mu,\nu,\alpha,\beta}$ satisfies all conditions in Definition 1.1 and $w_{\mu,\nu,\alpha,\beta} := \exp(-Q_{\mu,\nu,\alpha,\beta})$ is the desired weight.

We begin with estimates of $q_{\mu,\nu,\alpha,\beta}$. For $x \geqslant r$ we estimate $s(x),\ u(x),\ v(x)$ in (A.6). Then we use (1.2), (1.3), (1.4), (1.5) and (1.6). We see

$$s(x) \sim \log x, \quad s'(x) \sim \frac{1}{x}, \quad |s''(x)| \sim \frac{1}{x^2}, \tag{A.10}$$

$$u(x) \sim \log Q(x), \quad u'(x) \sim \frac{Q(x)}{Q'(x)} \leqslant CQ(x)^{\lambda-1}, \quad |u''(x)| \leqslant C(\frac{Q'(x)}{Q(x)})^2 \leqslant CQ(x)^{2(\lambda-1)} \tag{A.11}$$

and

$$v(x) \sim \log Q'(x) \leqslant C \log Q(x), \quad |v'(x)| \leqslant C\frac{Q''(x)}{Q'(x)} \leqslant C\frac{Q'(x)}{Q(x)} \leqslant CQ(x)^{\lambda-1},$$
$$|v''(x)| \leqslant C(\frac{Q''(x)}{Q'(x)})^2 \leqslant C(\frac{Q'(x)}{Q(x)})^2 \leqslant CQ(x)^{2(\lambda-1)}. \tag{A.12}$$

Therefore, by (A.10), (A.11) and (A.12) we have the followings. Let $x \geqslant r$.

$$\frac{q_{\mu,\nu,\alpha,\beta}(x)}{Q(x)} \leqslant C\frac{\log Q(x)}{Q(x)}, \ \frac{q'_{\mu,\nu,\alpha,\beta}(x)}{Q(x)} \leqslant C\frac{1}{Q(x)^{2-\lambda}}, \ \frac{q''_{\mu,\nu,\alpha,\beta}(x)}{Q(x)} \leqslant C\frac{1}{Q(x)^{3-2\lambda}}, \tag{A.13}$$

$$\frac{q_{\mu,\nu,\alpha,\beta}(x)}{Q'(x)} \leqslant C\frac{\log Q(x)}{Q'(x)} \leqslant C\frac{x\log Q(x)}{Q(x)}, \ \frac{q'_{\mu,\nu,\alpha,\beta}(x)}{Q'(x)} \leqslant C\frac{1}{Q(x)^{2-\lambda}},$$
$$\frac{q''_{\mu,\nu,\alpha,\beta}(x)}{Q'(x)} \leqslant C\frac{Q'(x)}{Q(x)^2} \leqslant C\frac{1}{Q(x)^{3-\lambda}}, \tag{A.14}$$

and

$$\frac{q_{\mu,\nu,\alpha,\beta}(x)}{Q''(x)} \leqslant C\frac{\log Q(x)}{Q''(x)} \leqslant C\frac{Q(x)\log Q(x)}{Q'(x)^2} \leqslant C\frac{x^2Q(x)\log Q(x)}{Q(x)^2} = C\frac{x^2\log Q(x)}{Q(x)},$$
$$\frac{q'_{\mu,\nu,\alpha,\beta}(x)}{Q''(x)} \leqslant C\frac{Q'(x)}{Q(x)Q''(x)} \leqslant C\frac{1}{Q'(x)} \leqslant C\frac{x}{Q(x)},$$
$$\frac{q''_{\mu,\nu,\alpha,\beta}(x)}{Q''(x)} \leqslant C\frac{1}{Q''(x)}(\frac{Q'(x)}{Q(x)})^2 \leqslant C\frac{Q(x)}{Q'(x)}\frac{Q(x)}{Q(x)^2} \leqslant C\frac{1}{Q(x)}. \tag{A.15}$$

Consequently, by (A.13), (A.14), (A.15) and $x^2 << Q(x)$ we have

$$\lim_{x\to\infty}\frac{q^{(i)}_{\mu,\nu,\alpha,\beta}(x)}{Q^{(j)}(x)} = 0, \quad i,j = 0,1,2. \tag{A.16}$$

Furthermore, by (A.7), (A.8) and (A.9), we also see that $x \in [r, 2r]$, then we have

$$|\frac{p^{(j)}_{\mu,\nu,\alpha,\beta}(x)}{Q(x)}| \leqslant \frac{C}{r^j}[\frac{\log Q(r)}{Q(r)} + \frac{r}{Q(r)^{2-\lambda}} + \frac{r^2}{Q(r)^{3-2\lambda}}],$$
$$|\frac{p^{(j)}_{\mu,\nu,\alpha,\beta}(x)}{Q'(x)}| \leqslant \frac{C}{r^j}[\frac{r\log Q(r)}{Q(r)} + \frac{r}{Q(r)} + \frac{r^2}{Q(r)^{2-\lambda}}],$$
$$|\frac{p^{(j)}_{\mu,\nu,\alpha,\beta}(x)}{Q''(x)}| \leqslant \frac{C}{r^j}[\frac{r^2\log Q(r)}{Q(r)} + \frac{r^2}{Q(r)} + \frac{r^2}{Q(r)}]$$

for $j = 0,1,2$. For $\Lambda > 1$ in (1.2), we take $\varepsilon > 0$ sufficiently small such that

$$\Lambda' := \Lambda(1-\varepsilon)/(1+\varepsilon) > 1. \tag{A.17}$$

By above estimates there exists $r = r_{\mu,\nu,\alpha,\beta} > 0$ such that $|x| \geqslant r$, $Q''(x) > 0$ and

$$1-\varepsilon \leqslant \frac{Q_{\mu,\nu,\alpha,\beta}(x)}{Q(x)}, \quad \frac{Q'_{\mu,\nu,\alpha,\beta}(x)}{Q'(x)}, \quad \frac{Q''_{\mu,\nu,\alpha,\beta}(x)}{Q''(x)} \leqslant 1+\varepsilon. \tag{A.18}$$

The inequality (A.18) means $w_{\mu,\nu,\alpha,\beta} \in \mathcal{F}(C^2+)$. We see that (A.3) and (A.4) follow as [9, pp.94-95].

Under the condition $w \in \mathcal{F}_\lambda(C^4+)$ we will show (A.5). We see

$$|s'''(x) \leqslant C\frac{1}{x^3}, \quad |u'''(x)|| \leqslant C|\frac{Q'(x)}{Q(x)}|^3, \quad |v'''(x)| \leqslant C|\frac{Q'(x)}{Q(x)}|^3.$$

Hence we have

$$\frac{|q'''(x)|}{Q(x)} \leqslant C\frac{1}{Q(x)^{4-3\lambda}}, \ \frac{|q'''(x)|}{Q'(x)} \leqslant C\frac{1}{Q(x)^{3-2\lambda}}, \ \frac{|q'''(x)|}{Q''(x)} \leqslant C\frac{1}{Q(x)^{2-\lambda}},$$
$$\frac{|q'''(x)|}{Q'''(x)} \leqslant C\frac{1}{Q(x)}.$$

Now, we take a polynomial

$$p_{\mu,\nu,\alpha,\beta}(x) = p_{\mu,\nu,\alpha,\beta,r}(x) = (2r-x)^4(ax^3+bx^2+cx+d)$$

such that

$$p_{\mu,\nu,\alpha,\beta}(r) = q_{\mu,\nu,\alpha,\beta}(r), \ p'_{\mu,\nu,\alpha,\beta}(r) = q'_{\mu,\nu,\alpha,\beta}(r),$$
$$p''_{\mu,\nu,\alpha,\beta}(r) = q''_{\mu,\nu,\alpha,\beta}(r), \ p'''_{\mu,\nu,\alpha,\beta}(r) = q'''_{\mu,\nu,\alpha,\beta}(r).$$

Now we set

$$Q_{\mu,\nu,\alpha,\beta}(x) := \begin{cases} Q(x), & \text{if } |x| \leqslant r; \\ Q(x) - q_{\mu,\nu,\alpha,\beta}(x) + p_{\mu,\nu,\alpha,\beta}(x), & \text{if } r < |x| \leqslant 2r; \\ Q(x) - q_{\mu,\nu,\alpha,\beta}(x), & \text{if } 2r < |x|. \end{cases}$$

We also see that $x \in [r, 2r]$, then we have

$$|\frac{p^{(j)}_{\mu,\nu,\alpha,\beta}(x)}{Q(x)}| \leqslant \frac{C}{r^j}[\frac{\log Q(r)}{Q(r)} + \frac{r}{Q(r)^{2-\lambda}} + \frac{r^2}{Q(r)^{3-2\lambda}} + \frac{r^3}{Q(r)^{4-3\lambda}}],$$
$$|\frac{p^{(j)}_{\mu,\nu,\alpha,\beta}(x)}{Q'(x)}| \leqslant \frac{C}{r^j}[\frac{r\log Q(r)}{Q(r)} + \frac{r}{Q(r)} + \frac{r^2}{Q(r)^{2-\lambda}} + \frac{r^3}{Q(r)^{3-2\lambda}}],$$
$$|\frac{p^{(j)}_{\mu,\nu,\alpha,\beta}(x)}{Q''(x)}| \leqslant \frac{C}{r^j}[\frac{r^2\log Q(r)}{Q(r)} + \frac{r^2}{Q(r)} + \frac{r^2}{Q(r)} + \frac{r^2}{Q(r)}]$$

for $j = 0, 1, 2, 3$. Therefore, there exists $r = r_{\mu,\nu,\alpha,\beta} > 0$ such that $|x| \geqslant r$, and

$$1-\varepsilon \leqslant \frac{Q_{\mu,\nu,\alpha,\beta}(x)}{Q(x)}, \ \ \frac{Q'_{\mu,\nu,\alpha,\beta}(x)}{Q'(x)}, \ \ \frac{Q''_{\mu,\nu,\alpha,\beta}(x)}{Q''(x)}, \frac{Q'''_{\mu,\nu,\alpha,\beta}(x)}{Q'''(x)} \leqslant 1+\varepsilon. \tag{A.19}$$

Consequently, we have (A.5). #

Theorem B. Let $\alpha_1, \ \alpha_2, ..., \ \alpha_k \in \mathbb{R}$ and let $w \in \mathcal{F}_\lambda(C^4+), \ 1 < \lambda < 4/3$. Then we have

$$T^{\alpha_1} w \sim w_{\alpha_1} \in \mathcal{F}(C^3+), \tag{A.20}$$

and

$$T^{\alpha_2} w_{\alpha_1} \sim w_{\alpha_1,\alpha_2} \sim w_{\alpha_1+\alpha_2} \in \mathcal{F}(C^3+),$$

generally, for $j = 0, 1, ..., k-1$

$$T^{\alpha_{k+1}} w_{\alpha_1,\alpha_2,...,\alpha_k} \sim w_{\alpha_1+\alpha_2+...+\alpha_{k+1}} \in \mathcal{F}(C^3+).$$

Proof. If $w \in \mathcal{F}_\lambda(C^4+)$, then we have $w_{\alpha_1} \in \mathcal{F}_\lambda(C^3+)$ by (A.5), and so

$$T^{-\alpha_1} w_{\alpha_1} \sim T^{-\alpha_1} T^{\alpha_1} w \sim w \in \mathcal{F}_\lambda(C^3+).$$

Here we note that there exists $r_1 > 0$ such that for $|x| \geqslant 2r_1$

$$w_{\alpha_1}(x) = \exp(-Q(x) + \alpha_1 \log T(x)),$$

and for some $r_2 \geqslant 2r_1$, if $|x| \geqslant 2r_2$, then we have

$$T^{-\alpha_1}(x)w_{\alpha_1}(x) = \exp(-Q(x) + \alpha_1 \log T(x) - \alpha_1 T(x)) = \exp(-Q(x)).$$

Consequently, we see

$$T^{\alpha_2}w_{\alpha_1} = T^{\alpha_1+\alpha_2}T^{-\alpha_1}w_{\alpha_1} \sim T^{\alpha_1+\alpha_2}w \sim w_{\alpha_1+\alpha_2} \in \mathcal{F}(C^3+).$$

We continue this method inductively, then we have

$$\begin{aligned} &T^{\alpha_{k+1}}w_{\alpha_1+\ldots+\alpha_k} = T^{\alpha_1+\ldots+\alpha_{k+1}}T^{-(\alpha_1+\ldots+\alpha_k)}w_{\alpha_1+\ldots+\alpha_k} \\ &\sim T^{\alpha_1+\ldots+\alpha_{k+1}}w \sim w_{\alpha_1+\ldots+\alpha_{k+1}} \in \mathcal{F}(C^3+). \quad \# \end{aligned}$$

We have the following.

Theorem C. ([4, Theorem 2.3]) Let $\nu \geqslant 0$ be an integer. Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{\nu+3}+)$, where $0 < \lambda < (\nu+3)/(\nu+2)$. Suppose that $f \in C^\nu(\mathbb{R})$ with

$$\lim_{|x|\to\infty} T^{1/4}(x)f^{(\nu)}(x)w(x) = 0. \quad (A.21)$$

Then there exists an absolute constant $C_\nu > 0$ which depends only on ν such that for $0 \leqslant k \leqslant \nu$ and $x \in \mathbb{R}$,

$$\begin{aligned} &|(f^{(k)}(x) - P^{(k)}_{n,f,w}(x))w(x)| \leqslant CT^{k/2}(x)E_{n-k}(w_{1/4}, f^{(k)}) \\ &\leqslant C_\nu T^{k/2}(x)(\frac{a_n}{n})^{r-k}E_{n-k}(w_{1/4}, f^{(\nu)}), \end{aligned}$$

where $T^{1/4}(x)w(x) \sim w_{1/4} \in \mathcal{F}(C^2+)$.

Now we can improve Theorem C.

Theorem D. Let $\nu \geqslant 0$ be an integer. Let

$$w = \exp(-Q) \in \mathcal{F}_\lambda(C^4+), \quad 0 < \lambda < 4/3. \quad (A.22)$$

Suppose that $f \in C^\nu(\mathbb{R})$ with

$$\|T^{1/4}f^{(\nu)}w\|_{L_\infty(\mathbb{R})} < \infty. \quad (A.23)$$

Then there exists an absolute constant $C_\nu > 0$ which depends only on ν such that for $0 \leqslant k \leqslant \nu$ and $x \in \mathbb{R}$,

$$\begin{aligned} &|(f^{(k)}(x) - P^{(k)}_{n,f,w}(x))w(x)| \leqslant CT^{k/2}(x)E_{n-k}(w_{1/4}, f^{(k)}) \\ &\leqslant C_\nu T^{k/2}(x)(\frac{a_n}{n})^{\nu-k}E_{n-\nu}(w_{1/4}, f^{(\nu)}), \end{aligned}$$

where $T^{1/4}(x)w(x) \sim w_{1/4} \in \mathcal{F}(C^2+)$.

To prove Theorem C we have used the following theorem (see [4, Proof of Theorem 2.3]).

Theorem E. ([9, Corollary 6.2]) Let $\nu \geqslant 0$ be an integer, $1 \leqslant p \leqslant \infty$ and

$$w \in \mathcal{F}_\lambda(C^{\nu+3}+), \ 1 < \lambda < \frac{\nu+3}{\nu+2}. \quad (A.24)$$

Then there exists a constant $C > 0$ such that for any $0 \leqslant k \leqslant \nu$, any integer $n \geqslant 1$ and any polynomial $P \in \mathcal{P}_n$,

$$\|P^{(k)}w\|_{L_p(\mathbb{R})} \leqslant C(\frac{n}{a_n})^k\|T^{k/2}Pw\|_{L_p(\mathbb{R})}. \quad (A.25)$$

If in Theorem E we can reduce the assumption (A.24) to the assumption (A.26), then we have the following.

Theorem F. Let $1 \leqslant p \leqslant \infty$ and

$$w \in \mathcal{F}_\lambda(C^4+),\ 1 < \lambda < \frac{4}{3}. \tag{A.26}$$

Then there exists a constant $C > 0$ such that for any $0 \leqslant k \leqslant \nu$, any integer $n \geqslant 1$ and any polynomial $P \in \mathcal{P}_n$,

$$\|P^{(k)}w\|_{L_p(\mathbb{R})} \leqslant C(\frac{n}{a_n})^k \|T^{k/2}Pw\|_{L_p(\mathbb{R})}. \tag{A.27}$$

Proof. In Theorem E we consider the case of $\nu = 2$, then we have the following.
Theorem G. ([9, Theorem 1.1]) Let $1 \leqslant p \leqslant \infty$, and let

$$w \in \mathcal{F}_\lambda(C^4+),\ 1 < \lambda < \frac{4}{3}. \tag{A.28}$$

Then there exists a constant $C > 0$ such that for any $0 \leqslant k \leqslant 2$, any integer $n \geqslant 1$ and any polynomial $P \in \mathcal{P}_n$,

$$\|P^{(k)}w\|_{L_p(\mathbb{R})} \leqslant C(\frac{n}{a_n})^k \|T^{k/2}Pw\|_{L_p(\mathbb{R})}. \tag{A.29}$$

In the proof of Theorem G, for $k = 1,\ w \in \mathcal{F}_\lambda(C^4+)$ we use $T^{1/2}w \sim w_{1/2} \in \mathcal{F}_\lambda(C^3+)$ (see (A.5)), and $k = 2,\ w \in \mathcal{F}_\lambda(C^4+)$ we use $T^{1/2}w_{1/2} \sim w_1 \in \mathcal{F}_\lambda(C^3+)$ (see (A.5)). If we consider the cases of $k = 3, 4, ...$, then with respect to w our assumption leaves $w \in \mathcal{F}_\lambda(C^4+)$, then $T^{1/2}w_{(k-1)/2} \sim w_{k/2} \in \mathcal{F}_\lambda(C^3+)$ (see (A.5)). In fact, Theorem B guarantees it.

Therefore, under the condition (A.28) we have the result for $k = 1, 2$ by Theorem G, and for $k \geqslant 3$ we see

$$T^{k/2}(x)w(x) \sim w_{k/2} \in \mathcal{F}(C^3+), \quad k = 3, 4, ...$$

by Theorem B. Consequently, we have Theorem F. #

Now we show Lemma 3.2, that is, Theorem D.

Proof of Lemma 3.2. Using the method of the proof of Theorem C, we can prove Theorem D applying Theorem F (see [4, Proof of Theorem 2.3]). Consequently, we have Lemma 3.2. #

