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Some Approaches for Fuzzy Multiobjective Programming Problems

Yves Mangongo Tinda[1] and Justin Dupar Kampempe Busili[2]

Department of Mathematics and Computer Science, University of Kinshasa, Kinshasa, D.R.Congo
Email: [1]ymangongo@aims.ac.tz, [1]yves.mangongo@unikin.ac.cd, [1]ymangongo@gmail.com,
[2]Justin-dupar.kampempe@Unikin.ac.cd

Abstract In this paper we discuss two approaches to bring a balance between effectiveness and efficiency while solving a multiobjective programming problem with fuzzy objective functions. To convert the original fuzzy optimization problem into deterministic terms, the first approach makes use of the Nearest Interval Approximation Operator (Approximation approach) for fuzzy numbers and the second one takes advantage of an Embedding Theorem for fuzzy numbers (Equivalence approach). The resulting optimization problem related to the first approach is handled via Karush-Kuhn-Tucker like conditions for Pareto Optimality obtained for the resulting interval optimization problem. A Galerkin like scheme is used to tackle the deterministic counterpart associated to the second approach. Our approaches enable both faithful representation of reality and computational tractability. They are thus in sharp contrast with many existing methods that are either effective or efficient but not both. Numerical examples are also supplemented for the sake of illustration.

Keywords: multiobjective programming, fuzzy numbers, nearest interval approximation, embedding theorem.

Introduction

A Multiobjective programming problem with fuzzy objective functions is a difficult optimization problem which arises in a wide array of useful applications in engineering [1], economics [2], finance [3], ecology [4], etc. Designing algorithms for this problem is therefore a relevant issue that has attracted intensive research activities (see e.g.[5] [6] [7]). A close look at the literature reveals that existing methods are either computationally tractable (efficient) or faithful in representing reality (effective) but not both. In this paper we present two approaches that care about both accuracy in representation of reality and tractability of the resulting problem. The first method approximates the original fuzzy problem using the Nearest Interval Approximation (NIA) Operator. It yields a tractable resulting interval Optimization problem for which we propose a technical solution through Karush-Kuhn-Tucker (KKT) like conditions for Pareto Optimality. These conditions have been obtained by making use of gH-differentiability [8]. The second method turns the original fuzzy problem into an equivalent deterministic one via an Embedding Theorem for fuzzy numbers [9]. The price to pay to pursue this equivalence avenue is high as the resulting optimization problem has infinite objective functions. We then resort to a Galerkin like scheme to solve this complex optimization problem. The paper is organized as follows: in Section 1 we present mathematical preliminaries that are needed in the sequel. In Section 2, we clearly formulate the optimization problem at hand. In Section 3, we discuss the approximation approach. Section 4 is devoted to the description of the equivalence approach. Finally in Section 5 we present some concluding remarks along with further developments in this field.

1 Preliminaries

1.1 gH-Differentiability of Interval Valued Functions

We denote by K_C the family of all bounded closed intervals in $\mathbb{R}$. For A, $B \in K_C$. The generalized Hukuhara difference (gH-difference) denoted by $A\Theta_{gH}B$ is defined by:

$$A\Theta_{gH}B = C \text{ iff} \begin{cases} \text{either} & A = B + C \\ \text{or} & B = A + (-1)C \end{cases}$$

The following are the two main order relations defined on K_C.

If $A = [a^L, a^U]$ and $B = [b^L, b^U]$ then we have:

$$A \leq_{LU} B \text{ if and only if } a^L \leq b^L \text{ and } a^U \leq b^U \tag{1}$$

$$A \leq_{LS} B \text{ if and only if } a^L \leq b^L \text{ and } a^S \leq b^S \tag{2}$$

where $a^S = a^U - a^L$ and $b^S = b^U - b^L$.

Proposition 1.1

$$A \leq_{LS} B \textit{ implies } A \leq_{LU} B$$

Consider a real interval T. The gH-derivative of an interval-valued function $F : T \longrightarrow K_C$ at t_0 is defined as:

$$F'(t_0) = \lim_{h \to 0} \frac{F(t_0 + h) \Theta_{gH} F(t_0)}{h}. \tag{3}$$

If $F'(t_0) \in K_C$ satisfying (1) exists. We say that F is gH-differentiable at t_0. If F is gH-differentiable at each point $t \in T$. We say that F is gH-differentiable on T. The generalization of the above definition of gH-derivative to interval-valued function defined on $\mathbb{R}^n$ is straightforward (see e.g. [10]).

1.2 Nearest Interval Approximation of Fuzzy Number

For the notion of fuzzy number, we refer the reader to Reference [9]. Let $\Im(R)$ be the space of fuzzy numbers. An interval approximation of a fuzzy number is an operator

$$C : \Im(R) \longrightarrow K_C$$

such that for $\tilde{a} \in \Im(R)$:

(i) $C(\tilde{a}) \subset Supp\, \tilde{a}$
(ii) $Core(\tilde{a}) \subset C(\tilde{a})$
(iii) $\forall \epsilon > 0,\ \exists \delta > 0, d_{\Im(R)}(\tilde{a}, \tilde{b}) < \delta \Longrightarrow H(C(\tilde{a}), C(\tilde{b})) < \epsilon$

where *Supp* stands for support; $d_{\Im(R)}$ and H are the metrics on $\Im(R)$ and K_C respectively.

A Nearest Interval Approximation (NIA) of a fuzzy number $\tilde{a}$ with respect to a metric $d_{\Im(R)}$, is an interval approximation of $\tilde{a}$, $C_d(\tilde{a})$ such that:

$$d_{\Im(R)}(C_d(\tilde{a}), \tilde{a}) \leq d_{\Im(R)}(C(\tilde{a}), \tilde{a});$$

for all interval approximation operator C.

Proposition 1.2 *If*

$$\begin{aligned} f^L : [0,\, 1] &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto f^L(\alpha) = \tilde{a}^L_\alpha \textit{ and} \\ f^U : [0,\, 1] &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto f^U(\alpha) = \tilde{a}^U_\alpha, \end{aligned}$$

then

$$C_d(\tilde{a}) = \left[\int_0^1 f^L(\alpha) d\alpha, \int_0^1 f^U(\alpha) d\alpha \right].$$

For the proof of this proposition we refer the reader to [11].

1.3 Embedding Theorem for Fuzzy Numbers

In the sequel we assume that $\mathcal{F}_{cc}(\mathbb{R})$ is the space of fuzzy numbers with compact support and $\tilde{C}[0,\,1]$ is the space of real-valued functions on $[0,1]$ such that:

- f is left continuous for any $t \in [0,\,1]$ and right continuous at 0.
- f has a right limit for any $t \in [0,\,1]$

It is worth noting that in this paper, $\mathcal{F}_{cc}(\mathbb{R})$ is endowed with the following order: for $\tilde{a}, \tilde{b} \in F_C$, $\tilde{a} \leq \tilde{b}$ iff $\tilde{a}^{\alpha} \leq \tilde{b}^{\alpha}$ for all α, where $\tilde{m}^{\alpha}$ is the α-level set of $\tilde{m}$.

Theorem 1.3 [9]
The following map is isomorphic and isometric.

$$\begin{aligned} \mathit{\Pi} : \mathcal{F}_{cc}(R) &\longrightarrow \tilde{C}([0,\,1]) \times \tilde{C}([0,\,1]) \\ \tilde{a} &\longrightarrow (\tilde{a}^L(\alpha),\, \tilde{a}^U(\alpha)) \end{aligned}$$

where $\tilde{a}^L(\alpha) = \tilde{a}^L_{\alpha}$, and $\tilde{a}^L_{\alpha}$, $\tilde{a}^U_{\alpha}$ are the lower and upper endpoints of $\tilde{a}^{\alpha}$ (the α-level of $\tilde{a}$) respectively.

2 Problem Formulation

The problem under consideration in this paper is the optimizing several fuzzy objective functions under crisp constraints. Without loss of generality to restrict ourselves to deterministic constraints as the literature is rich of approaches for converting fuzzy constraints into deterministic ones (see e.g[12]). More formally we consider a problem of the type:

$$(P1) \begin{cases} \max(\tilde{f}_1(x),\, \tilde{f}_2(x), \cdots, \tilde{f}_k(x)) \\ x \in X = \{x \in \mathbb{R}^n / g_j(x) \leq 0\ ,\ j = 1, \cdots m\} \end{cases}$$

where $\tilde{f}_i(x)$, $i = 1, 2, \cdots, k$ are fuzzy number valued functions in $\mathbb{R}^n$. This problem is of common occurrence in many real-life applications, including scheduling nursing resources, evaluation of urban policy, forest management, corporate financial management, portfolio selection etc.

3 Approximation Approach for Dealing with $(\boldsymbol{P}1)$

This approach has been presented in [16]. We present it here in a way to motivate its coupling with the approach based by considering an equivalent determinist problem.

3.1 Deterministic Surrogate of $(\boldsymbol{P}1)$

To find a solution of $(P1)$, we find its approximation by the following interval multiobjective program:

$$(P_2) \begin{cases} \min(F_1(x),\, F_2(x), \cdots, F_k(x)) \\ x \in X \end{cases}$$

where $F_i(x) = [f_i^L(x),\, f_i^U(x)]$, $i = 1,\, 2, \cdots,\, k$ stands for the nearest interval approximation (NIA) of $\tilde{f}_i(x)$.

3.2 Solutions Concept

Definition 3.1 *$x^* \in X$ is said to be a LU Pareto optimal solution of (P_2) if there is no $x \in X$ such that $F_i(x) \leq_{LU} F_i(x^*)$, $\forall i$ with at least one $l \in 1, 2, \cdots k$ such that $F_l(x) <_{LS} F_l(x^*)$. If $\leq_{LS}$ is considered instead of $\leq_{LU}$ then x^* is an LS-Pareto optimal solution of (P_2).*

Proposition 3.2 [4]
If x^* is LU-Pareto optimal solution of (P_2) then x^* is LS-Pareto optimal solution of (P_2).

Proof
Suppose that x^* is LU-Pareto optimal solution of (P_2) and not LS-Pareto optimal solution of (P_2). Then there is $x \in X$ such that $F_i(x) \leq_{LS} F_i(x^*)$ $\forall i$ with at least one $l \in 1, 2, \cdots k$ such that $F_l(x) \leq_{LS} F_l(x^*)$. By Proposition 2.1, we can say that there exists $x \in X$ such that $F_i(x) \leq_{LU} F_i(x^*)$ $\forall i$ and $F_l(x) <_{LU} F_l(x^*)$ for some l. But this contradicts the fact that x^* is LU-Pareto optimal solution of (P_2) and we are done.

3.3 Karush-Kuhn-Tucker Type Pareto Optimality Conditions for (P_2)

In this section we present some Karush-Kuhn-Tucker type Pareto optimality conditions for problem (P_2). The conditions are based on gH-differentiability of interval-valued functions.

Theorem 3.3 *Consider (P_2) and assume that the functions*

$$F_i : \mathcal{F}_{cc}^n \longrightarrow K_c \, (i = 1, \cdots, k)$$

are continuously gH-differentiable at x^*.

If $f_i^L + f_i^U$ $(i = 1, \cdots, k)$ are convex functions and if there are $\lambda_i > 0, (i = 1, \cdots, k)$ and such that $\mu_j^* \geq 0$, $j = 0, \cdots, m$:

(i)

$$\sum_{i}^{k} \lambda_i^* \nabla (f_i^L + f_i^U)(x^*) + \mu_j^* \nabla g_j(x^*) = 0$$

(ii) $\mu_j^* g_j(x^*) = 0, \, \forall j = 0, \cdots m$

then x^* is an LU-Pareto optimal solution for (P_2) .

Proof
We define the real-valued functions $h_i(x) = \lambda_i^* [f_i^L + f_i^U](x)$; $i = 1, \cdots, m$.

Since by hypothesis, $F_i(x)$, $i = 1, \cdots, k$ are convex functions we have that f_i^L and f_i^U are convex for all i and then $f_i^L + f_i^U$ are also convex for all i. Therefore h_i $(i = 1, \cdots, m)$ are convex functions.

Since $F_i(x)$, $i = 1, \cdots$, m are gH-differentiable at x^*, we have h_i, $i = 1, \cdots k$ are differentiable at x^* (see [8]).

As $\nabla h_i(x^*) = \lambda_i^* \nabla (f_i^L + f_i^U)(x^*)$, we have from (i) and (ii) that:

$$\sum_{i=1}^{k} \nabla h_i(x^*) + \sum_{i=1}^{k} \mu_j^* \nabla g_j(x^*) = 0$$

and

$$\mu_j^* \nabla g_j(x^*) = 0 \text{ for all } j \in \{0, 1, \cdots, m\} .$$

By Karush-Khun-Tucker for multiobjective (real valued) functions, x^* is Pareto optimal for the multiobjective program:

$$(P_3) \begin{cases} \min(h_1(x), h_2(x), \cdots, h_k(x)) \\ x \in X \end{cases} .$$

Suppose now that x^* is not an LU-Pareto optimal solution for (P_2). Then there is $x \in X$ such that:

$$\begin{aligned} F_1(x) &\leq_{LU} F_1(x^*) \\ &\vdots \\ F_k(x) &\leq_{LU} F_k(x^*) \end{aligned}$$

with $F_l(x) \neq F_l(x^*)$ for some $l \in \{1, 2, \cdots, k\}$.

This is tantamount to say that there is $x \in X$ such that for all i, we have either:

$$\begin{cases} f_i^L(x) < f_i^L(x^*) \\ f_i^U(x) \leq f_i^U(x^*) \end{cases}$$

or

$$\begin{cases} f_i^L(x) \leq f_i^L(x^*) \\ f_i^U(x) < f_i^U(x^*) \end{cases}$$

or

$$\begin{cases} f_i^L(x) < f_i^L(x^*) \\ f_i^U(x) < f_i^U(x^*) \end{cases}.$$

Therefore we have that for all i

$$h_i(x) < h_i(x^*).$$

This means that x^* is not Pareto optimal solution for (P_3). This is a contradiction. Hence x^* is an LU-Pareto optimal solution for (P_2).

Corollaire 3.4 *If the assumptions of Theorem 3.3 hold then x^* is an LS-Pareto optimal solution for (P_2).*

Proof
This comes from Proposition 3.2 and Theorem 3.3.

Definition 3.5 *An LU (or LS)-Pareto optimal solution of (P_2) is called satisficing solution of $(P1)$.*

3.4 Algorithm for Finding a Satisficing Solution of $(P1)$

From the above discussion we can derive the following algorithm for finding a satisficing solution of $(P1)$
Algorithme 1

- Start
- input: k,m
  objectives functions: $\tilde{f}_1(x), \cdots, \tilde{f}_k(x)$
- Constraint functions: $g_1(x), \cdots, g_m(x)$

1. Finding nearest interval approximation of : $\tilde{f}_1(x), \cdots, \tilde{f}_k(x)$
   (a) Put $l = 1$
   (b) Repeat until break
   If $l \leq k$
   Write NIA of $\tilde{f}_l(x) = C_d(\tilde{f}_l(x))$; where $C_d(\tilde{a})$ is obtained as in Proposition 1.2.
   Else, break and go to (c).
2. Finding a satisficing solution of $(P1)$
   (c) Fix $\lambda_i^* > 0$, $(i = 1, \ldots, m)$
   (d) Choose $\mu_j^* \geq 0$ $(j = 0, \cdots, m)$ such that the system (S) is compatible

$$(S) \begin{cases} \sum_{i=1}^{k} \lambda_k^* \nabla \left(f_i^L + f_i^U\right)(x) + \mu_j^* \nabla g_j(x^*) = 0 \\ \mu_j^* g_j(x) = 0,\ j = \{1, \ldots, m\} \end{cases}$$

   (e) Solve (S), if the solution exits, let x^* be its solution, then go to (g) else go to (f)
   (f) If such $\left\{\mu_j^*\right\}_j$, defined in (d), cannot be found go to (h)
   (g) Print : << x^* is a satisficing solution of $(P1)$ >> Go to (i).
   (h) Print : << There is not satisficing solution for $(P1)$ >>
   (i) Stop.

3.5 Numerical Example

In this section we illustrate the proposed method in a numerical example. Consider the following multi-objective program:

$$(P_4)\begin{cases} \min\left(\tilde{c}_1^1 x_1+\tilde{c}_2^1 x_2, \tilde{c}_1^2 x_1+\tilde{c}_2^2 x_2\right) \\ \text{ subject to} \\ -x_1-x_2\leq -6 \\ -2x_1-x_2\leq -9 \\ x_1,\ x_2\geq 0 \end{cases}$$

where $\tilde{c}_j^k$ $(k=1,2; j=1,2)$ are fuzzy numbers with the membership functions given below:

$$\mu_{\tilde{c}_1^1}(x)=\begin{cases} 2x-1 & \text{for } x\in[0.5,1] \\ -2x+3 & \text{for } x\in[1,1.5] \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_{\tilde{c}_2^1}(x)=\begin{cases} 5x-9 & \text{for } x\in[1.8,2] \\ -x+3 & \text{for } x\in[2,3] \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_{\tilde{c}_1^2}(x)=\begin{cases} \frac{1}{2}x & \text{for } x\in[0,2] \\ -x+3 & \text{for } x\in[2,3] \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_{\tilde{c}_2^2}(x)=\begin{cases} x & \text{for } x\in[0,1] \\ -\frac{1}{2}x+\frac{3}{2} & \text{for } x\in[1,3] \\ 0 & \text{elsewhere} \end{cases}$$

Using Proposition 1.2, we obtain the near interval approximations of these fuzzy numbers as follows. $\tilde{c}_1^1,\ \tilde{c}_2^1,\ \tilde{c}_1^2,\ \tilde{c}_2^2$:

$$\begin{aligned} c_d\left[\tilde{c}_1^1\right] &= \left[\int_0^1 \frac{(\alpha+1)}{2}d\alpha, \int_0^1 \frac{(3-\alpha)}{2}d\alpha\right] \\ &= [0.75, 1.25] \\ c_d\left[\tilde{c}_2^1\right] &= \left[\int_0^1 \frac{(\alpha+9)}{5}d\alpha, \int_0^1 (3-\alpha)d\alpha\right] \\ &= [1.9, 2.5] \\ c_d\left[\tilde{c}_1^2\right] &= \left[\int_0^1 2\alpha d\alpha, \int_0^1 (3-\alpha)d\alpha\right] \\ &= [1, 2.5] \\ c_d\left[\tilde{c}_2^2\right] &= \left[\int_0^1 \alpha d\alpha, \int_0^1 (3-2\alpha)d\alpha\right] \\ &= [0.5, 2] \end{aligned}$$

These intervals contain the core and are included in the support of the corresponding fuzzy number.

The counterpart of the system (S) for (P_4) is as follows:

$$(\S)\begin{cases} \lambda_1\left[\begin{pmatrix}0.75\\1.9\end{pmatrix}+\begin{pmatrix}1.25\\25\end{pmatrix}\right]+\lambda_2\left[\begin{pmatrix}1\\0.5\end{pmatrix}+\begin{pmatrix}2.5\\2\end{pmatrix}\right]+\mu_1\begin{pmatrix}-1\\-2\end{pmatrix}-\mu_2=0 \\ \mu_1\left(-x_1-x_2+6\right)=0 \\ \mu_2\left(-2x_1-x_2+9\right)=0 \\ \lambda_1>0, \lambda_2>0, \mu_1\geq 0, \mu_2\geq 0. \end{cases}$$

So if we find $\lambda_1^* > 0, \lambda_2^* > 0$, $\mu_1^* \geq 0, \mu_2^* \geq 0$ and $x^* = (x_1^*, x_2^*)$ such that $(\lambda_1^*, \lambda_2^*, \mu_1^*, \mu_2^*, x_1^*, x_2^*)$ verifies the above system, then (x_1^*, x_2^*) is a satisficing solution of (P_4). For $\lambda_1^* = \lambda_2^* = 1$, the first equation of (§) reads

$$\begin{cases} \mu_1 + \mu_2 = 0.75 + 1.25 + 1.0 + 2.5 = 5.5 \\ 2\mu_1 + \mu_2 = 1.9 + 2.5 + 0.5 + 2 = 6.9. \end{cases}$$

which yields $\mu_1^* = 1.4 > 0$, $\mu_2^* = 4.1 > 0$.

As $x^* = (3, 3)$ is the solution of the Cramerian system

$$\begin{cases} x_1 + x_2 = 6 \\ 2x_1 + x_2 = 9 \end{cases}$$

We see that the vector $(1, 1, 1.4, 4.1, 3, 3)$ is a solution of (§).

Therefore $x^* = (3, 3)$ is a satisficing solution of (P_4).

4 Approach Based on Considering an Equivalent Deterministic Counterpart

4.1 Equivalent Deterministic Counterpart of $(P1)$

Consider the map Π (see Theorem 1.3) and replace $(P1)$ by:

$$(P1)' \begin{cases} \max(\Pi(f_1(x),\, f_2(x), \cdots, f_k(x))) \\ x \in X \end{cases}$$

The following result, bridges (as expected) the gap between $(P1)$ and $(P1)'$.

Theorem 4.1 *$x^* \in X$ is a Pareto optimal solution for $(P1)$ if and only if it is a Pareto optimal solution for $(P1)'$.*

Proof

Assume x^* is a Pareto optimal solution for $(P1)$. Then there is no $x \in X$ such that:

$$\tilde{f}_i(x^*) \leq \tilde{f}_i(x), \ \text{ for all } i \in \{1, \cdots, k\}$$

and

$$\tilde{f}_l(x^*) < \tilde{f}_l(x), \ \text{ for some } l \in \{1, \cdots, k\}$$

As Π is order preserving (See Theorem 1.3), it is tantamount to say that there is no $x \in X$ such that:

$$\pi\tilde{f}_i(x^*) \leq \pi\tilde{f}_i(x) \ \text{ for all } i$$

and

$$\pi\tilde{f}_l(x^*) < \pi\tilde{f}_l(x) \ \text{ for some } l$$

This means that x^* is a Pareto optimal solution for $(P1)'$.

By Theorem 5.1 $(P1)$ and $(P1)'$ are equivalent. Moreover $(P1)'$ can be written as:

$$(P1)'' \begin{cases} \max\left(\left[f_{1\alpha}^L(x),\, f_{1\alpha}^U(x)\right], \cdots, \left[f_{k\alpha}^L(x), f_{k\alpha}^U(x)\right]\right) \\ \text{ for all } \alpha \in I = (0,\, 1]; \\ x \in X \end{cases}$$

For simplicity, let us define the real-valued functions $g_{i\alpha} : i = 1,\, 2, \cdots, 2k$ as follows

$$\begin{array}{rl} g_{1\alpha}(x) = f_{1\alpha}^L(x) \ ; & g_{2\alpha}(x) = f_{1\alpha}^U(x) \\ g_{3\alpha}(x) = f_{2\alpha}^L(x) \ ; & g_{4\alpha}(x) = f_{2\alpha}^U(x) \\ \cdots \ \ \cdots \ ; & \cdots \ \ \cdots \\ g_{2l-1,\alpha}(x) = f_{l\alpha}^L(x) \ ; & g_{2l,\alpha}(x) = f_{l\alpha}^U(x) \\ \cdots \ \ \cdots \ ; & \cdots \ \ \cdots \\ g_{2k-1,\alpha}(x) = f_{1\alpha}^k(x) \ ; & g_{2k,\alpha}(x) = f_{2k\alpha}^U(x) \end{array}$$

If we consider the order relation $\leq_{LU}$, $(P1)''$ reads

$$(P1)_1''' \begin{cases} \max(f_{1\alpha}^L(x),\, f_{1\alpha}^U(x), \cdots, f_{k\alpha}^L(x), f_{k\alpha}^U(x)) \\ \forall \alpha \in I = [0,\, 1]\,; \\ x \in X \end{cases}$$

or

$$(P1)_2''' \begin{cases} \max(g_{1\alpha}(x),\, g_{2\alpha}(x)(x), \cdots, g_{2k-1\alpha}(x),\, g_{2k\alpha}(x)) \\ \text{ for all } \alpha \in [0,\, 1] = I; \\ x \in X \end{cases}$$

or

$$(P1)_3''' \begin{cases} \max(g_{i\alpha}(x)) \ \forall (i,\, \alpha) \in P \times I \\ x \in X \end{cases}$$

where $P = \{1,\, 2, \cdots, k\}$ and $I = [0,\, 1]$.

4.2 An Approximation Method for Dealing with $(P1)_3'''$

Analysis Let us consider a finite subset I_m of $I = [0,\, 1]$ say $I_m = \{\alpha_1,\, \alpha_2, \cdots, \alpha_m\}$.

Let $\omega_1,\, \omega_2, \cdots, \omega_m$ be real-valued functions on [0, 1] such that:

(i) $\omega_j(\alpha) \geq 0$ for $(j,\, \alpha) \in \{1,\, 2,\, \cdots, m\} \times I$

(ii) $\omega_j(\alpha_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Define now an operator that associates to $g_{i\alpha}$, $Kg_{i\alpha} : X \longrightarrow \mathbb{R}$ given by

$$Kg_{i\alpha}(x) = \sum_{j=1}^{k} \omega_j(\alpha) g_{i\alpha_j}(x)$$

K is called a positive interpolating operator with modes $\alpha_1,\, \alpha_2, \cdots, \alpha_m$.

Consider now the following mathematical programs:

$$(P1)^{\text{iv}} \begin{cases} \max(Kg_{i\alpha}(x));\, \forall (i,\, \alpha) \in P \times I \\ x \in X \end{cases}$$

$$(P1)^{\text{v}} \begin{cases} \max(g_{i\alpha_j}(x));\, \forall (i,\, \alpha_j) \in P \times I_m \\ x \in X \end{cases}$$

The following result bridges a gap between $(P1)^{\text{iv}}$ and $(P1)^{\text{v}}$ in terms of Pareto optimality.

Theorem 4.2 *$x^* \in X$ is Pareto optimal for $(P1)^{iv}$ if and only if it is also Pareto optimal for $(P1)^{v}$.*

Proof

(a) Let us first prove the necessary condition.

Assume x^* is Pareto optimal solution for $(P1)^{\text{iv}}$ and not Pareto optimal solution for $(P1)^{\text{v}}$. Then there is no $x \in X$ such that

$$(Kg_{i\alpha}(x^*)) \leq (Kg_{i\alpha}(x)) \text{ for all } (i, \alpha) \in P \times I \tag{4}$$

and

$$(Kg_{l\alpha}(x^*)) < (Kg_{l\bar{\alpha}}(x)) \text{ for some } l \in \{1, \cdots, 2k\} \text{ and } \bar{\alpha} \in (0, 1]\,. \tag{5}$$

In the same time we have, by the fact that x^* is not efficient for $(P1)^{\text{v}}$, there is $x \in X$ such that

$$g_{i\alpha_j}(x^*) \leq g_{i\alpha_j}(x) \text{ for all } (i, \alpha_j) \in P \times I_m \tag{6}$$

and

$$g_{i\alpha_s}(x^*) < g_{i\alpha_s}(x) \text{ for some } (l, \alpha_s) \in P \times I_m. \tag{7}$$

Let us now choose α arbitrarily in $(0,1]$.

As $\omega_j(\alpha) \geq 0$ and $\omega_j(\alpha_i) = \delta_{ij}$. We have by (3) and (4) that:

$$\sum_{j=1}^{m} \omega_j(\alpha) \left[g_{i\alpha_j}(x^*) - g_{i\alpha_j}(x) \right] \leq 0, \ \forall i \in \{1, \cdots, 2k\}.$$

This means that

$$\sum_{j=1}^{m} \omega_j(\alpha) g_{i\alpha_j}(x^*) \leq \sum_{j=1}^{m} \omega_j(\alpha) g_{i\alpha_j}(x), \ \forall i \in \{1, \cdots, 2k\}.$$

Since α has been chosen arbitrarily, there is $x \in X$ such that

$$(Kg_{i\alpha})(x^*) \leq (Kg_{i\alpha})(x) \ \forall\, (i,\alpha) \in P \times I$$

This contradicts (4).

Therefore x^* should be Pareto optimal solution for $(P1)^{\mathrm{v}}$.

(b) Let us now prove the sufficient condition.

Assume x^* is Pareto optimal for $(P1)^{\mathrm{v}}$ and not Pareto optimal for $(P1)^{\mathrm{iv}}$.

By the Pareto optimality of x^* for $(P1)^{\mathrm{v}}$, there is no $x \in X$ such that :

$$g_{i\alpha_j}(x^*) \leq g_{i\alpha_j}(x), \ \forall\, (i, \alpha_j) \in P \times I_m$$

and

$$g_{l\alpha_s}(x^*) < g_{l\alpha_s}(x), \ \text{for some } (l, \alpha_s) \in P \times I_m.$$

This means that:

$$g_{i\alpha_j}(x^*) - g_{i\alpha_j}(x) > 0, \ \forall x \in X, \ \forall\, (i, \alpha_j) \in P \times I_m \tag{8}$$

and

$$g_{l\alpha_s}(x^*) - g_{l\alpha_s}(x) \geq 0, \ \forall x \in X. \tag{9}$$

Now by virtue of the non Pareto optimality of x^* for $(P1)^{\mathrm{iv}}$, there is $x \in X$ such that

$$(Kg_{i\alpha})(x^*) \leq (Kg_{i\alpha})(x) \text{ for all } (i,\alpha) \in P \times I_m. \tag{10}$$

and

$$(Kg_{l\bar{\alpha}})(x^*) < (Kg_{l\bar{\alpha}})(x) \text{ for some } (l,\bar{\alpha}) \in P \times I_m. \tag{11}$$

We take α arbitrarily in I.

Since $\omega_j(\alpha) \geq 0$, $\forall\, (j,\alpha) \in P \times I$, by (8) and (9), for all $x \in X$ we have:

$$\sum_{j=1}^{m} \omega_j(\alpha) \left[g_{i\alpha_j}(x) - g_{i\alpha_j}(x^*) \right] = \sum_{j/\omega_j(\alpha)\neq 0} \left[g_{i\alpha_j}(x) - g_{i\alpha_j}(x^*) \right] < 0$$

$$\forall\, x \in X, \ \forall (i,\alpha) \in P \times I.$$

So

$$\sum_{j=1}^{m} \omega_j(\alpha) \left[g_{i\alpha_j}(x) - g_{i\alpha_j}(x^*) \right] < 0 \ \forall\, x \in X, \ \forall (i,\alpha) \in P \times I.$$

This means that, we have

$$\sum_{j=1}^{m} \omega_j(\alpha) g_{i\alpha_j}(x) < \sum_{j=1}^{m} \omega_j(\alpha) g_{i\alpha_j}(x^*) \quad \forall x \in X, \forall (i,\alpha) \in P \times I.$$

i.e.

$$(Kg_{i\alpha})(x) < (Kg_{i\alpha})(x^*) \quad \forall x \in X, \forall (i,\alpha) \in P \times I.$$

This is in contradiction with (10) thus x^* should be Pareto optimal solution for $(P_1)^{\mathrm{iv}}$, and that completes the proof.

Remark Theorem 5.2 tells us that solving the optimization problem $(P1)^{\text{v}}$, with a finite number of objective functions, is equivalent to solving the problem that interests us: $(P1)'''$ (having infinitely objective functions) where $g_{i\alpha}(x)$ is replaced by $(Kg_{i\alpha})(x)$. This algorithm focus on a finite multi objective programming problem $(P1)^{\text{v}}$ while dealing with an optimization problem with infinitely many objective function.

Caution should be exercised in keeping $|(Kg_{i\alpha})(x) - g_{i\alpha}(x)|$ small, in a way to minimize the approximation errors.

This is done by keeping the roughness of the grid as small as possible [17].

In what follows, l_i stands for the number of points in the grid at i^{th} discretization.

We start with two points 0 and 1.

The sequence $\{l_i\}_{i\geq 2}$ is obtained by:

$$l_i = l_{i-1} + 2^{i-2};\ i \geq 2$$

We stop when $h = \frac{1}{2(l_i-1)} < \epsilon$, where ϵ is the upper bound fixed for the roughness of the grid h.

Description of an Algorithm for Solving $(P1)_3'''$

Step 1 Fix an acceptable upper bound for the roughness of the grid h say $\epsilon > 0$.
Step 2 Read data of $(P1)_3^{'''}$.
Step 3 Put $i = 1$, $l_i = 2$, $\alpha_{li_1} = 0$, $\alpha_{li_2} = 1$, $S_{li_1} = \{\alpha_{li_1}, \alpha_{li_2}\}$
Step 4 Compute

$$h = \max_{\alpha\in I} \min_{1\leq j\leq S} |\alpha - \alpha_{l_{ij}}|,$$

where S is such that $\alpha_{l_{ij}} = 1$.
Step 5 Check whether $h \leq \epsilon$.
If yes go to Step 6 otherwise take a finer dicretization of I, S_{l+1}. Put $l_i = l_{i+1}$ and go to Step 9.
Step 6 Write $(P1)^{\text{v}}$
Step 7 Find a Pareto optimal solution of $(P1)^{\text{v}}$
Step 8 Print x^* is a satisficing solution of $(P1)_3^{'''}$
Step 9 Stop

5 Numerical Example

Consider the following multiobjective optimization problem:

$$\begin{cases} \max(\tilde{c}^1 x, \tilde{c}^2 x) \\ \text{ subject to} \\ -x_1 + 2x_2 \leq 4 \\ x_1 + x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{cases}$$

where $\tilde{c}^1 = (\tilde{c}_1^1, \tilde{c}_2^1)$ and $\tilde{c}^2 = (\tilde{c}_1^2, \tilde{c}_2^2), \tilde{c}_j^i \in \{1, 2\}$ are triangular fuzzy numbers with the following membership functions.

$$\mu_{\tilde{c}_1^1}(x) = \begin{cases} 2x - 1 & \text{if } x \in [0.5, 1] \\ -2x + 3 & \text{if } x \in [1, 1.5] \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_{\tilde{c}_2^1}(x) = \begin{cases} 5x - 9 & \text{if } x \in [1.8, 2] \\ -x + 3 & \text{if } x \in [2, 3] \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_{\tilde{c}_1^2}(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, 2] \\ -x + 3 & \text{for } x \in [2, 3] \\ 0 & \text{elsewhere} \end{cases}$$

We have to find a Pareto optimal solution of the counterpart of $(P1)^{\text{v}}$ corresponding to (P_2) . That is we have to find a Pareto optimal solution of the following multi objective program:

$$(P1)' \begin{cases} \max(g_{1\alpha_j}(x),\, g_{2\alpha_j}(x), \cdots, g_{4\alpha_j}(x)),\ \forall \alpha_j \in S_{l_3} \\ -x_1 + 2x_2 \leq 4 \\ x_1 + x_2 \leq 6 \\ x_1,\, x_2 \geq 0 \end{cases}$$

where

$$\begin{cases} g_{1\alpha_j}(x) = \tilde{c}^{1}_{1}{}^{L}_{\alpha_j} x_1 + \tilde{c}^{1}_{2}{}^{L}_{\alpha_j} x_2 \\ g_{2\alpha_j}(x) = \tilde{c}^{1}_{1}{}^{U}_{\alpha_j} x_1 + \tilde{c}^{1}_{2}{}^{U}_{\alpha_j} x_2 \\ g_{3\alpha_j}(x) = \tilde{c}^{2}_{1}{}^{L}_{\alpha_j} x_1 + \tilde{c}^{2}_{2}{}^{L}_{\alpha_j} x_2 \\ g_{4\alpha_j}(x) = \tilde{c}^{2}_{1}{}^{U}_{\alpha_j} x_1 + \tilde{c}^{2}_{2}{}^{U}_{\alpha_j} x_2 \end{cases}$$

where $S_{l_3} = (0,\, 0.25,\, 0.5,\, 0.75,\, 1)$

The below table give us objective functions of $(P_2)^v$

Table 1. Objective functions of the discretized problem

$\alpha_j / f_{i\alpha_j}$	$g_{1\alpha_j}$	$g_{2\alpha_j}$	$g_{3\alpha_j}$	$g_{4\alpha_j}$
0	$0.5x_1 + 1.8x_2$	$1.5x_1 + 3x_2$	-	$3x_1 + 3x_2$
0.25	$0.625x_1 + 1.8x_2$	$1.375x_1 + 2.75x_2$	$0.5x_1 + 0.5x_2$	$2.75x_1 + 2.5x_2$
0.5	$0.75x_1 + 1.9x_2$	$1.25x_1 + 2.5x_2$	$x_1 + 0.5x_2$	-
0.75	$0.875x_1 + 1.95x_2$	$1.125x_1 + 2.25x_2$	$1.5x_1 + 0.75x_2$	$2.25x_1 + 1.5x_2$
1	$x_1 + 2x_2$	-	$2x_1 + x_2$	-

After having removed redundant objective functions, we can solve the following single objective optimization problem to obtain a Pareto optimal solution of $(P_2)^v$

$$(P_3) \begin{cases} \max \sum_i \sum_j \lambda_{ij} g_{i\alpha_j}(x) \\ -x_1 + 2x_2 \leq 4 \\ x_1 + x_2 \leq 6 \\ x_1,\, x_2 \geq 0 \end{cases}$$

Taking $\lambda_{ij} = \frac{1}{16}\ \forall (i,\, j)$ corresponding to a non zero and non redundant objective function, and solving (P_3) using LINGO software, we obtain $x^* = (2.6,\, 3.3)$ the approximate Pareto optimal solution for the equivalent deterministic program of the original fuzzy multi objective program (P_2).

6 Concluding Remarks

Fuzzy multi objective programming problems are encountered in a wide spectrum of applications in engineering, economic and finance [10],[13],[14]. Neither the option of squeezing arbitrarily conflictual objective functions into a single one, nor that of replacing blindly imprecise data by fixed ones, is appropriate in this context. Such strategies would leave no other option to the model but to produce meaningless outcomes.

In this paper, we have proposed two approaches that help to strike a balance between faithful representation of reality and computational tractability in a lower case for fuzzy. This is in stark contrast with real deffuzification approaches [13] or possibility, necessity based approaches [14]. These approaches are respectively efficient but not effective, and effective but not efficient. Our first approach makes use of the nearest interval approximation operator to approximate the original problem by interval optimization program. We have derived some Karush-Kuhn-Tucker like conditions for Pareto optimality based on gH-differentiability, that helped us solve the above mentioned interval program.

The main idea behind our second approach is to take advantage of an Embedding Theorem for fuzzy numbers in a way to put the original problem into equivalent deterministic terms. The price for this effectiveness is quite high as the resulting deterministic program is computationally demanding. We then described a Galerkin like scheme for tackling this complex deterministic optimization problem. Kirby [15] argued that the main objections against Operations Research (OR) techniques are as follows:

1. OR techniques ignore managerial needs (pervesion criticism).
2. OR methods have already been used wherever they were needed (obsolescence criticism).
3. Management needs have evolved and are more complex than those which OR caters for (inadequacy criticism).
4. OR's practice has been misguided and has undermined the confidence managers had in it (coun-ter-performance criticism).

In this paper, we have made a modest contribution towards remedying to some extent the above mentioned perversion and inadequacy objections. In the future, we will explore the possibility of combining the best features of the two approaches discussed in this paper within a Decision Support System. Another line for further developments in this field is the of design Intelligent Hybrid Algorithms for tackling situations where randomness and fuzziness co-occur in a multi objective programming setting.

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