

Stability of Contact Discontinuity with General Perturbation for the Compressible Navier-Stokes Equations with Reaction Diffusion

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Abstract In this article, for the compressible Navier-Stokes equations which have reaction diffusion, the stability of contact discontinuities is considered. The new characteristic for the flow is appearance of the divergence between energy gained and lost because of the reaction. In the energy equations, the term related to the mass fraction of the reactant leads to new technical problem. To solve this problem, in terms of the solutions, a new system should be set up. Using the anti-derivative method and the elaborated energy method, we obtain that as long as the general perturbation of the initial datum plane and the strength of the contact wave are properly small, the contact wave is nonlinear and stable. As a byproduct, we can establish the convergence velocity of contact wave.

Keywords: Compressible Navier-Stokes, reaction diffusion, Cauchy problem, contact waves, convergence rates.

1 Introduction

In this paper, we consider the stability of the following compressible Navier-Stokes equations for a reacting mixture.

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, \theta)_x = \left(\frac{\mu u_x}{v}\right)_x, \\ \left(e + \frac{u^2}{2}\right)_t + (p(v, \theta)u)_x = \left(\frac{\mu u u_x}{v}\right)_x + \left(\frac{\kappa \theta_x}{v}\right)_x + \lambda \varphi z, \\ z_t = \left(\frac{dz_x}{v^2}\right)_x - \varphi z, \end{cases} \quad (1.1)$$

where $x \in \mathbb{R}$ is the Lagrangian space variable, $t \in \mathbb{R}^+$ the time variable and the primary dependent variables are the specific volume $v = v(t, x) > 0$, the velocity $u = u(t, x)$, the absolute temperature $\theta = \theta(t, x) > 0$ and the mass fraction of the reactant $z = z(t, x)$. The last term $\lambda \varphi z$ on the right hand side of the energy equation (1.1)₃ represents the difference of the rate of energy gained by the product and that of energy lost to the reactant as a result of the reaction. The positive constants d and λ are the specific diffusion coefficient and the difference in the heat between the reactant and the product, respectively. The reaction rate function $\varphi = \varphi(\theta)$ is defined, from the Arrhenius law [5], by

$$\varphi(\theta) = \begin{cases} 0, & 0 \leq \theta \leq \theta_I, \\ K\theta^\beta \exp\left(-\frac{A}{\theta}\right), & \theta > \theta_I, \quad \beta \geq 0, \end{cases}$$

where the positive constants K and A are the coefficients of the rate of the reactant and the activation energy. This function describes that combustion will occur when the temperature of the given fluid particle rises above the ignition temperature θ_I with $\theta_I \geq 0$. As a result, the reactant ($z = 1$) is transformed to the product ($z = 0$) via an irreversible reaction governed by the function $\varphi(\theta)$.

The positive constants μ and κ denote the viscosity coefficient and the heat conduction coefficient, respectively. The pressure p and the internal energy e are given by the state equations:

$$p(v, \theta) = \frac{R\theta}{v}, \quad e = \frac{R}{\gamma - 1}\theta,$$

where R and γ are the positive constants. We present the initial data for the system (1.1) as follows.

$$(v_0, u_0, \theta_0, z_0)(x) \rightarrow (v_{\pm}, 0, \theta_{\pm}, 0), \quad \text{as } x \rightarrow \pm\infty, \tag{1.2}$$

where $v_{\pm} > 0$ and $\theta_{\pm} > \theta_I$ are constants and

$$p_- \triangleq \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} \triangleq p_+. \tag{1.3}$$

Now we will give an overview on the study for (1.1). The global existence and asymptotic behavior of solutions to (1.1) were obtained by Chen [2] with discontinuous reacting rate functions. In [4], Chen, Hoff and Trivisa proved the existence and dynamic behavior of discontinuous solutions with discontinuous initial data. The existence theory of global solutions to (1.1) on unbounded domains was established by Li [22]. The long time behavior toward rarefaction waves for the Cauchy problem to (1.1) was shown by Xu and Feng in [39]. Recently, Peng [28] studied the stability of viscous contact wave for the Cauchy problem under the zero mass condition on the initial perturbation. When the effect of radiation is taken into account, the total pressure p of the gas is expressed as $p = \frac{R\theta}{v} + \frac{a}{3}\theta^4$ which includes a four-order radiative part through the Stefan-Boltzmann law, c.f., e.g., Mihalas and Mihalas [27]. In such a case, the global existence and uniqueness of solutions of the Cauchy problem was achieved by Liao and Zhao [25] with constant viscosity coefficients and by He et al. [10] with temperature dependent viscosity coefficients. The existence and uniqueness theory for the initial boundary value problem for the one-dimensional model (1.1) has been well established, see [3,7,8,9,20,21,24,29,32,34]. For the study of the three-dimensional case, we refer the readers to [6,23,30,33,36,42] and the references therein. In this paper, we study the stability toward contact waves for solutions to the system (1.1) provided that the general perturbation of the initial data is suitably small. When $z = 0$, the system (1.1) is reduced to the compressible Navier-Stokes equations with heat conduction. Indeed, there are many studies providing insight into the contact wave phenomena in the development of the mathematical theory for compressible Navier-Stokes equations, see [12,11,13,14,15,16,18,19,31,35,38,40,41] and the references therein. As far as we know, a first work of the stability toward contact waves for solutions to systems of viscous conservation laws was represented by Xin in [37], where the metastability of a weak contact discontinuity for the compressible Euler system with uniform viscosity was investigated. The local stability of the contact discontinuities for a class of general system of viscous conservation laws with artificial viscosity was studied by Liu and Xin [26]. For the compressible Navier-Stokes system, the main difficulty is that the viscosity matrix for Navier-Stokes equations is only semi-positive definite. Huang, Matsumura and Xin [15] used the anti-derivative method to obtain not only the stability of the viscous contact waves for solutions to the compressible Navier-Stokes system, but also the convergence rate $(1+t)^{-\frac{1}{4}}$ under the zero mass condition on the initial perturbation. By using a weighted energy method, Huang, Wang and Wang in [17] obtained a better decay rate $(1+t)^{-\frac{5}{8}+C\sqrt{\delta}}$ for the wave strength $\delta > 0$ suitably small. Recently, this convergence rate was improved to $(1+t)^{-\frac{5}{8}} \ln^{\frac{1}{2}}(2+t)$ by Yang in [40]. It is worth recalling that the major assumption in the stability theory in [15] that the initial perturbation has zero excessive mass is a crucial constraint. This constraint of the zero mass condition on the initial perturbation was removed by Huang, Xin and Yang in [18]. Motivated by [18], the main purpose of this paper is to get rid of the constraint difficulty for initial excessive mass stated in [28] and obtain the stability and convergence rate for the viscous contact waves.

Now, we focus on the asymptotical behavior of solutions to (1.1) according to the far field states (1.2) and (1.3) without the zero mass condition on the initial perturbation. First of all, we recall the contact wave $(\bar{v}, \bar{u}, \bar{\theta})(x, t)$ for the compressible Navier-Stokes equations defined in [15]. For a corresponding Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, \theta)_x = 0, \\ \left(e + \frac{u^2}{2} \right)_t + (pu)_x = 0, \end{cases} \tag{1.4}$$

a contact discontinuity takes the form

$$(\bar{V}, \bar{U}, \bar{\Theta})(x, t) = \begin{cases} (v_-, 0, \theta_-), & x < 0, \\ (v_+, 0, \theta_+), & x > 0, \end{cases}$$

if the positive constants v_{\pm} and θ_{\pm} satisfy (1.3). We assume that the contact wave profile $(\bar{v}, \bar{u}, \bar{\theta}, \bar{z})$ exists as a smooth function and is given as follows

$$\bar{v} = \frac{R\Theta}{p_+}, \quad \bar{u} = \frac{Ra}{p_+\Theta}\Theta_x, \quad \bar{\theta} = \Theta - \frac{\gamma-1}{2R}\bar{u}^2, \quad \bar{z} = 0. \tag{1.5}$$

Here Θ is a smooth function which is determined by a nonlinear diffusion equation

$$\Theta_t = a \left(\frac{\Theta_x}{\Theta} \right)_x, \quad a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0, \tag{1.6}$$

with the boundary conditions

$$\Theta(\pm\infty, t) = \theta_{\pm}. \tag{1.7}$$

From [1], the problem (1.6) and (1.7) admits a unique self-similar solution $\theta(t, x) = \Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}$. Furthermore, $\Theta(\xi)$ is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_+ < \theta_-$. On the other hand, there exists some positive constant $\tilde{\delta}$, such that for $\delta = |\theta_+ - \theta_-| \leq \tilde{\delta}$, Θ satisfies

$$(1+t)^{\frac{k}{2}} |\partial_x^k \Theta| + |\Theta - \theta_{\pm}| \leq c_1 \delta e^{-\frac{c_0 x^2}{1+t}}, \quad k \geq 1, \quad \text{as } |x| \rightarrow \infty, \tag{1.8}$$

where c_0 and c_1 are two positive constants depending only on θ_- and $\tilde{\delta}$. It is straightforward to check that $(\bar{v}, \bar{u}, \bar{\theta})$ satisfies

$$\|(\bar{v} - \bar{V}, \bar{u} - \bar{U}, \bar{\theta} - \bar{\Theta})\|_{L^p} = O(\kappa^{1/(2p)})(1+t)^{1/(2p)}, \quad p \geq 1,$$

which means the nonlinear diffusion wave $(\bar{v}, \bar{u}, \bar{\theta})(x, t)$ approximates the contact discontinuity $(\bar{V}, \bar{U}, \bar{\Theta})(x, t)$ to the Euler system (1.3) in L^p norm, $p \geq 1$ on any finite time interval as the heat conductivity coefficient κ tends to zero. And it is easy to check that the viscous contact wave $(\bar{v}, \bar{u}, \bar{\theta})$ satisfies the system

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \bar{u}_t + \bar{p}_x = \mu \left(\frac{\bar{u}_x}{\bar{v}} \right)_x + R_{1x}, \\ \left(\bar{e} + \frac{\bar{u}^2}{2} \right)_t + (\bar{p}\bar{u})_x = \kappa \left(\frac{\bar{\theta}_x}{\bar{v}} \right)_x + \left(\frac{\mu}{\bar{v}} \bar{u} \bar{u}_x \right)_x + R_{2x}, \end{cases} \tag{1.9}$$

where

$$\begin{aligned} R_1 &= \left(\frac{\kappa(\gamma-1)}{\gamma R} - \mu \right) \frac{\bar{u}_x}{\bar{v}} + \bar{p} - p_+ \\ &= O(\delta)(1+t)^{-1} e^{-\frac{\theta_{\pm} x^2}{4\alpha(1+t)}} \quad \text{as } |x| \rightarrow \infty, \end{aligned} \tag{1.10}$$

$$\begin{aligned} R_2 &= \left(\frac{\kappa(\gamma-1)}{\gamma R} - \mu \right) \frac{\bar{u}\bar{u}_x}{\bar{v}} + (\bar{p} - p_+)\bar{u} \\ &= O(\delta)(1+t)^{-\frac{3}{2}} e^{-\frac{\theta_{\pm} x^2}{4\alpha(1+t)}} \quad \text{as } |x| \rightarrow \infty. \end{aligned} \tag{1.11}$$

Denote

$$m(x, t) = \left(v, u, \theta + \frac{\gamma-1}{2R}u^2 \right)^t, \quad \bar{m}(x, t) = \left(\bar{v}, \bar{u}, \bar{\theta} + \frac{\gamma-1}{2R}\bar{u}^2 \right)^t. \tag{1.12}$$

Let

$$A(v, u, \theta) = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p}{v} & 0 & \frac{R}{v} \\ -\frac{(\gamma-1)pu}{Rv} & \frac{\gamma-1}{R}p & \frac{(\gamma-1)u}{v} \end{pmatrix}. \tag{1.13}$$

The first eigenvalue of $A(v_-, 0, \theta_-)$ is $\lambda_1^- = -\sqrt{\frac{\gamma p_-}{v_-}}$ with right eigenvector

$$r_1^- = \left(-1, \lambda_1^-, \frac{\gamma - 1}{R} p_-\right)^T. \tag{1.14}$$

The third eigenvalue of $A(v_+, 0, \theta_+)$ is $\lambda_3^+ = -\sqrt{\frac{\gamma p_+}{v_+}}$ with right eigenvector

$$r_3^+ = \left(-1, \lambda_3^+, \frac{\gamma - 1}{R} p_+\right)^T. \tag{1.15}$$

The vectors $r_1^-, m_+ - m_- = (v_+ - v_-, 0, \theta_+ - \theta_-)^T$ and r_3^+ are linearly independent in \mathbb{R}^3 . Similar to [18], the integral $\int_{-\infty}^{+\infty} (m(x, 0) - \bar{m}(x, 0)) dx$ can be distributed as follows

$$\int_{-\infty}^{+\infty} (m(x, 0) - \bar{m}(x, 0)) dx = \bar{\theta}_1 r_1^- + \bar{\theta}_2 (m_+ - m_-) + \bar{\theta}_3 r_3^+, \tag{1.16}$$

with some constants $\bar{\theta}_i, i = 1, 2, 3$.

Set

$$\tilde{m}(x, t) = \bar{m}(x + \bar{\theta}_2, t) + \bar{\theta}_1 \theta_1 r_1^- + \bar{\theta}_3 \theta_3 r_3^+, \tag{1.17}$$

where

$$\begin{aligned} \theta_1(x, t) &= \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x - \lambda_1^-(1+t))^2}{4(1+t)}}, \\ \theta_3(x, t) &= \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x - \lambda_3^+(1+t))^2}{4(1+t)}}, \end{aligned} \tag{1.18}$$

satisfying

$$\theta_{1t} + \lambda_1^- \theta_{1x} = \theta_{1xx}, \quad \theta_{3t} + \lambda_3^+ \theta_{3x} = \theta_{3xx}, \tag{1.19}$$

and $\int_{-\infty}^{+\infty} \theta_i(x, t) dx = 1$ for $i = 1, 2, 3$.

Denote

$$\tilde{m}(x, t) = \left(\tilde{v}, \tilde{u}, \tilde{\theta} + \frac{\gamma - 1}{2R} \tilde{u}^2\right)^T(x, t), \tag{1.20}$$

with

$$\begin{aligned} \tilde{v}(x, t) &= \bar{v}(x + \bar{\theta}_2, t) - \bar{\theta}_1 \theta_1 - \bar{\theta}_3 \theta_3, \\ \tilde{u}(x, t) &= \bar{u}(x + \bar{\theta}_2, t) + \lambda_1^- \bar{\theta}_1 \theta_1 + \lambda_3^+ \bar{\theta}_3 \theta_3, \\ \tilde{\theta}(x, t) &= \bar{\theta}(x + \bar{\theta}_2, t) + \frac{\gamma - 1}{2R} \bar{u}^2(x + \bar{\theta}_2, t) \\ &\quad + \frac{\gamma - 1}{R} p_+ (\bar{\theta}_1 \theta_1 + \bar{\theta}_3 \theta_3) - \frac{\gamma - 1}{2R} \bar{u}^2. \end{aligned} \tag{1.21}$$

Then

$$\begin{aligned} &\int_{-\infty}^{+\infty} (m(x, 0) - \tilde{m}(x, 0)) dx \\ &= \int_{-\infty}^{+\infty} (m(x, 0) - \bar{m}(x, 0)) dx + \int_{-\infty}^{+\infty} (\bar{m}(x, 0) - \tilde{m}(x, 0)) dx \\ &= \bar{\theta}_2 (m_+ - m_-) + \int_{-\infty}^{+\infty} (\bar{m}(x, 0) - \bar{m}(x + \bar{\theta}_2, 0)) dx = 0. \end{aligned} \tag{1.22}$$

In the following, we assume $\bar{\theta}_2 = 0$ for simplification. Then,

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = \tilde{R}_{1x}, \\ \tilde{u}_t + \tilde{p}_x = \mu \left(\frac{\tilde{u}_x}{\tilde{v}}\right)_x + \tilde{R}_{2x}, \\ \left(\tilde{e} + \frac{\tilde{u}^2}{2}\right)_t + (\tilde{p}\tilde{u})_x = \kappa \left(\frac{\tilde{\theta}_x}{\tilde{v}}\right)_x + \left(\frac{\mu\tilde{u}\tilde{u}_x}{\tilde{v}}\right)_x + \tilde{R}_{3x}, \end{cases} \tag{1.23}$$

where

$$\tilde{R}_1 = -\bar{\theta}_1\theta_{1x} - \bar{\theta}_3\theta_{3x}, \tag{1.24}$$

$$\begin{aligned} \tilde{R}_2 = & R_1 + \mu \left(\frac{\bar{u}_x}{\bar{v}} - \frac{\tilde{u}_x}{\tilde{v}}\right) + (\lambda_1^- \bar{\theta}_1\theta_{1x} + \lambda_3^+ \bar{\theta}_3\theta_{3x}) \\ & + (\tilde{p} - \bar{p} - (\lambda_1^-)^2 \bar{\theta}_1\theta_1 - (\lambda_3^+)^2 \bar{\theta}_3\theta_3), \end{aligned} \tag{1.25}$$

and

$$\begin{aligned} \tilde{R}_3 = & R_2 + \kappa \left(\frac{\bar{\theta}_x}{\bar{v}} - \frac{\tilde{\theta}_x}{\tilde{v}}\right) + \mu \left(\frac{\bar{u}\bar{u}_x}{\bar{v}} - \frac{\tilde{u}\tilde{u}_x}{\tilde{v}}\right) + p_+ (\bar{\theta}_1\theta_{1x} + \bar{\theta}_3\theta_{3x}) \\ & + (\tilde{p}\tilde{u} - \bar{p}\bar{u} - p_+ \lambda_1^- \bar{\theta}_1\theta_1 - p_+ \lambda_3^+ \bar{\theta}_3\theta_3). \end{aligned} \tag{1.26}$$

Denote the perturbation around $(\bar{v}, \bar{u}, \bar{\theta}, 0)$ by

$$\phi(x, t) = v - \bar{v}, \quad \psi(x, t) = u - \bar{u}, \quad \zeta(x, t) = \theta - \bar{\theta}, \tag{1.27}$$

and set

$$\Phi(x, t) = \int_{-\infty}^x \phi(y, t) dy, \quad \Psi(x, t) = \int_{-\infty}^x \psi(y, t) dy, \tag{1.28}$$

$$\bar{W}(x, t) = \int_{-\infty}^x \left(e + \frac{|u|^2}{2} - \bar{e} - \frac{|\tilde{u}|^2}{2} + \lambda z \right) (y, t) dy. \tag{1.29}$$

Now we are in a position to state the main result in this paper.

Theorem 1.1. *Let $(\tilde{v}, \tilde{u}, \tilde{\theta})(x, t)$ be defined in (1.21) and $\delta = |\theta_+ - \theta_-|$. Then there exist positive constants δ_0 and ϵ , such that if $\delta \leq \delta_0$ and the initial data $(v_0, u_0, \theta_0, z_0)$ satisfy*

$$\|(\Phi, \Psi, \bar{W})(\cdot, t = 0)\|_{H^2(\mathbb{R})} \leq \epsilon, \tag{1.30}$$

and

$$\|z_0(x)\|_{H^1(\mathbb{R})} \leq \epsilon, \quad 0 \leq z_0(x) \leq 1, \tag{1.31}$$

then the system admits a unique global solution $(v, u, \theta, z)(x, t)$ satisfying

$$(\Phi, \Psi, \bar{W}) \in C(0, +\infty; H^2(\mathbb{R})), \quad z \in C(0, +\infty; H^1(\mathbb{R})), \tag{1.32}$$

$$\phi \in L^2(0, +\infty; H^1(\mathbb{R})), \quad (\psi, \zeta, z) \in L^2(0, +\infty; H^2(\mathbb{R})). \tag{1.33}$$

Furthermore, the solution satisfies

$$\|(v - \tilde{v}, u - \tilde{u}, \theta - \tilde{\theta})\|_{L^\infty(\mathbb{R})} \leq C(\epsilon + \delta_0^{\frac{1}{2}})(1 + t)^{-\frac{1}{4}}, \tag{1.34}$$

and

$$\|z\|_{L^\infty(\mathbb{R})} \leq C(\epsilon + \delta_0^{\frac{1}{2}})e^{-Ct}. \tag{1.35}$$

Remark 1.1. *Compared with the compressible Navier-Stokes equations, the term related to z in the energy equation (1.1)₃ leads to new technical difficulties, such as deducing the $L_t^1 L_x^1$ -norm estimates of the term $\lambda\phi z\zeta$. One of the main reasons is that the anti-derivative technique can not be directly applied to the system (1.1). To overcome the difficulties, our way is to construct a new perturbation system (2.1) about some modes Φ, Ψ and \bar{W} .*

Notations. Throughout this paper, c and C denote the generic positive constants depending only on the initial data and physical coefficients but independent of time t . The norms in the Sobolev spaces $H^m(\mathbb{R})$ is denoted by $\|\cdot\|_{H^m}$ for integer $m \geq 0$. In particular, for $m = 0$, we will simply use $\|\cdot\|$.

The rest of this paper is organized as follows. In Section 2, we reformulate the system (1.1). Then, the local in time existence and some *a priori* estimates for the solutions are established in Section 3. In Section 4, we obtain the decay rates of solutions and hence the global existence theory of Theorem 1.1 by using the standard continuity method.

2 Reformulated System

Since $(\phi, \psi) = (\Phi, \Psi)_x$ and $\frac{R}{\gamma-1}\zeta + \lambda z + \frac{1}{2}|\Psi_x|^2 + \tilde{u}\Psi_x = \bar{W}_x$, we have

$$\begin{cases} \Phi_t - \Psi_x = -\tilde{R}_1, \\ \Psi_t + p - \tilde{p} = \frac{\mu}{v}u_x - \frac{\mu}{\tilde{v}}\tilde{u}_x - \tilde{R}_2, \\ \bar{W}_t + pu - \tilde{p}\tilde{u} = \frac{\kappa}{v}\theta_x - \frac{\kappa}{\tilde{v}}\tilde{\theta}_x + \frac{\mu}{v}uu_x - \frac{\mu}{\tilde{v}}\tilde{u}\tilde{u}_x - \tilde{R}_3 + \frac{\lambda d}{v^2}z_x, \end{cases} \tag{2.1}$$

For simplify, a new variable is introduced as follows:

$$W = \frac{\gamma-1}{R}(\bar{W} - \tilde{u}\Psi). \tag{2.2}$$

It follows that

$$\zeta + \frac{\lambda(\gamma-1)}{R}z = W_x - Y, \quad \text{with } Y = \frac{\gamma-1}{R} \left(\frac{1}{2}\Psi_x^2 - \tilde{u}_x\Psi \right). \tag{2.3}$$

Then, the system (2.1) can be rewritten as

$$\begin{cases} \Phi_t - \Psi_x = -\tilde{R}_1, \\ \Psi_t - \frac{p_+}{v}\Phi_x + \frac{R}{v}W_x - \frac{\mu}{v}\Psi_{xx} = G_1, \\ \frac{R}{\gamma-1}W_t + p_+\Psi_x - \frac{\kappa}{v}W_{xx} = G_2, \end{cases} \tag{2.4}$$

where the right hand side terms of (2.4) are

$$G_1 = \left(\frac{\mu}{v} - \frac{\mu}{\tilde{v}} \right) u_x + \frac{R}{\tilde{v}}Y + J_1 - \tilde{R}_2 + \frac{\lambda(\gamma-1)}{\tilde{v}}z, \tag{2.5}$$

$$\begin{aligned} G_2 = & \left(\frac{\kappa}{v} - \frac{\kappa}{\tilde{v}} \right) \theta_x + \frac{\mu u_x}{v}\Psi_x - \tilde{R}_3 - \tilde{u}_t\Psi + \tilde{u}\tilde{R}_2 + J_2 \\ & - \frac{\kappa}{\tilde{v}}Y_x - \frac{\lambda\kappa(\gamma-1)}{R\tilde{v}}z_x + \frac{\lambda d}{v^2}z_x, \end{aligned} \tag{2.6}$$

with

$$\begin{aligned} J_1 = & \frac{\tilde{p} - p_+}{\tilde{v}}\Phi_x - \left[p - \tilde{p} + \frac{\tilde{p}}{\tilde{v}}\Phi_x - \frac{R}{\tilde{v}}(\theta - \tilde{\theta}) \right] \\ = & O(1)(\Phi_x^2 + W_x^2 + Y^2 + z^2 + |\tilde{u}|^4), \end{aligned} \tag{2.7}$$

$$J_2 = (p_+ - p)\Psi_x = O(1)(\Phi_x^2 + \Psi_x^2 + W_x^2 + Y^2 + z^2 + |\tilde{u}|^4). \tag{2.8}$$

3 Some *a priori* Estimates

In this section, we establish some *a priori* estimates bounded with respect to time. As usual, the global existence of solutions will be obtained by combining to the local existence result with some *a priori* estimates and then employing the standard continuity argument. In what follows, we first show the local in time existence of solutions to the Cauchy problem (1.1) and (1.2).

Proposition 3.1 (Local existence). *Under the assumptions of Theorem 1.1, for any constant $M_0 > 0$, there exists positive constants \bar{C}_1 and $T_1 = T_1(M_0) > 0$ such that if $\|(\phi_0(\cdot), \psi_0(\cdot), \zeta_0(\cdot), z_0(\cdot))\|_{H^1} \leq M_0$, then the Cauchy problem (1.1) and (1.2) admits a unique solution (ϕ, ψ, ζ, z) satisfying*

$$\begin{aligned} (\phi, \psi, \zeta, z) &\in C([0, T_1], H^1), \quad (\phi_x, z) \in L^2([0, T_1], L^2), \\ (\psi_x, \zeta_x, z_x) &\in L^2([0, T_1], H^1), \end{aligned} \tag{3.1}$$

and

$$\sup_{t \in [0, T_1]} \|(\phi, \psi, \zeta, z)(t, \cdot)\|_{H^1}^2 \leq \bar{C}_1 M_0. \tag{3.2}$$

Remark 3.1. *The proof of Proposition 3.1 can be completed by using standard iteration arguments. We refer for instance to [2]. We omit the details for brevity.*

To prove Theorem 1.1, we need to close the following *a priori* assumption:

$$N(T) := \sup_{0 \leq t \leq T} \{ \|(\Phi, \Psi, W)\|_{L^\infty}^2 + \|(\phi, \psi, \zeta, z)\|_{H^1}^2 \} \leq \varepsilon_0^2, \tag{3.3}$$

where ε_0 is a positive small constant. By (1.16), it is obvious that $|\bar{\theta}_1| + |\bar{\theta}_3| \leq C\varepsilon_0$. The generalized profile components \tilde{v} , $\tilde{\theta}$ have positive lower and upper bounds, which are only determined by initial data. If ε_0 is suitably small, then we have

$$\frac{1}{2}\tilde{v} \leq v = \tilde{v} + \phi \leq 2\tilde{v} \quad \text{and} \quad \frac{1}{2}\tilde{\theta} \leq \theta = \tilde{\theta} + \zeta \leq 2\tilde{\theta}. \tag{3.4}$$

Consequently, we will use

$$0 < c \leq \tilde{v}, v, \tilde{\theta}, \theta \leq C, \tag{3.5}$$

directly here and after. Similar to Lemma 2 in [2], we have

$$0 \leq z(x, t) \leq 1. \tag{3.6}$$

Remark 3.2. *To close the low order estimates of the solutions, the $L_t^\infty L_x^2$ -norm estimate of (Φ, Ψ, W) is allowed to grow at the rate $(1+t)^{\frac{1}{4}}$. However, this growth of the low order energy norm is compensated by the decay in the energy norm of higher order derivatives. The elementary energy method based on the anti-derivative argument introduced in [18] is used to obtain some desired energy estimates bounded with respect to time and hence the global existence.*

3.1 Low Order Estimates on (Φ, Ψ, W)

In this subsection, we establish the $L_t^\infty L_x^2$ -norm estimates of (Φ, Ψ, W, Φ_x) and the $L_t^2 L_x^2$ -norm estimates of (Φ_x, Ψ_x, W_x) by using the classical energy method.

Lemma 3.1. *Let $\bar{\delta} = \delta + \theta_1 + \theta_3$. It holds that*

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{p_+}{2} \Phi^2 + \frac{\tilde{v}}{2} \Psi^2 + \frac{R^2}{2(\gamma-1)p_+} W^2 \right) dx + \frac{1}{2} \int_{\mathbb{R}} \left(\mu \Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}} W_x^2 \right) dx \\ &\leq (C\bar{\delta} + \frac{1}{4})(1+t)^{-1} (\|\Phi\|^2 + \|\Psi\|^2 + \|W\|^2) + C(1+t)\|z\|^2 + C\bar{\delta}(1+t)^{-\frac{1}{2}} \\ &\quad + C(\bar{\delta} + \varepsilon_0)(\|\Phi_x\|^2 + \|z\|^2 + \|(\phi, \psi, \zeta)_x\|^2). \end{aligned} \tag{3.7}$$

Proof. Multiplying the equations (2.4)₁, (2.4)₂ and (2.4)₃ by $p_+\Phi$, $\tilde{v}\Psi$ and $\frac{R}{p_+}W$, respectively, we have

$$\begin{aligned} &\left(\frac{p_+}{2} \Phi^2 + \frac{\tilde{v}}{2} \Psi^2 + \frac{R^2}{2(\gamma-1)p_+} W^2 \right)_t + \mu \Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}} W_x^2 \\ &= \frac{1}{2} \tilde{v}_t \Psi^2 - \tilde{R}_1 p_+ \Phi - \left(\frac{R\kappa}{p_+\tilde{v}} \right)_x W W_x + \tilde{v} G_1 \Psi + \frac{R}{p_+} W G_2 + (\dots)_x, \end{aligned} \tag{3.8}$$

where the notation $(\dots)_x$ represents the term in the conservative form. By integration (3.8) with respect to x in \mathbb{R} , we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left(\frac{p_+}{2} \Phi^2 + \frac{\tilde{v}}{2} \Psi^2 + \frac{R^2}{2(\gamma-1)p_+} W^2 \right) + \int_{\mathbb{R}} \left(\mu \Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}} W_x^2 \right) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} \tilde{v}_t \Psi^2 dx - \int_{\mathbb{R}} \tilde{R}_1 p_+ \Phi dx - \int_{\mathbb{R}} \left(\frac{R\kappa}{p_+\tilde{v}} \right)_x W W_x dx \\ & \quad + \int_{\mathbb{R}} \tilde{v} G_1 \Psi dx + \int_{\mathbb{R}} \frac{R}{p_+} W G_2 dx. \end{aligned} \tag{3.9}$$

Now we will estimate the right hand side terms of (3.9) as follows. By using (1.5), (1.6), (1.8), (1.18) and (1.21), we have

$$\frac{1}{2} \int_{\mathbb{R}} |\tilde{v}_t| \Psi^2 dx \leq C\bar{\delta}(1+t)^{-1} \|\Psi\|^2. \tag{3.10}$$

Noticing (1.24) and using (1.18) and the Cauchy-Schwarz inequality, we get

$$\int_{\mathbb{R}} |\tilde{R}_1 \Phi| dx \leq C\bar{\delta}(1+t)^{-1} \|\Phi\|^2 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.11}$$

Similarly, by (1.8), (1.18) and the Cauchy-Schwarz inequality, we obtain

$$\left| \int_{\mathbb{R}} \left(\frac{R\kappa}{p_+\tilde{v}} \right)_x W W_x dx \right| \leq C\bar{\delta}(1+t)^{-1} \|W\|^2 + C\bar{\delta} \|W_x\|^2. \tag{3.12}$$

In the following, we show the estimates of the last two terms on the right hand side of (3.9). It is clear that

$$\begin{aligned} \left| \int_{\mathbb{R}} \tilde{v} G_1 \Psi dx \right| &\leq \int_{\mathbb{R}} |\tilde{v}| \left| \left(\frac{\mu}{v} - \frac{\mu}{\tilde{v}} \right) u_x \right| |\Psi| dx + \int_{\mathbb{R}} R|Y| |\Psi| dx \\ &\quad + \int_{\mathbb{R}} |\tilde{v}| |J_1| |\Psi| dx + \int_{\mathbb{R}} |\tilde{v}| |\tilde{R}_2| |\Psi| dx + \lambda(\gamma-1) \int_{\mathbb{R}} |z| |\Psi| dx \\ &:= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.13}$$

For I_1 , by (1.8), (1.18), (1.21) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}} \left| \frac{\mu \Phi_x}{v} \right| |\psi_x| |\Psi| dx + \int_{\mathbb{R}} \left| \frac{\mu \Phi_x}{v} \right| |\tilde{u}_x| |\Psi| dx \\ &\leq C(\bar{\delta} + \varepsilon_0) \|\Phi_x\|^2 + C\varepsilon_0 \|\psi_x\|^2 + C\bar{\delta}(1+t)^{-1} \|\Psi\|^2. \end{aligned} \tag{3.14}$$

By the assumption (3.3) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_2 &\leq C \int_{\mathbb{R}} |\Psi_x|^2 |\Psi| dx + C \int_{\mathbb{R}} |\tilde{u}_x| |\Psi|^2 dx \\ &\leq C\varepsilon_0 \|\Psi_x\|^2 + C\bar{\delta}(1+t)^{-1} \|\Psi\|^2. \end{aligned} \tag{3.15}$$

For I_3 , it is obvious that

$$I_3 \leq \int_{\mathbb{R}} \left| \frac{\tilde{p} - p_+}{\tilde{v}} \Phi_x \right| |\Psi| dx + \int_{\mathbb{R}} \left| p - \tilde{p} + \frac{\tilde{p}}{\tilde{v}} \Phi_x - \frac{R}{\tilde{v}} (\theta - \tilde{\theta}) \right| |\Psi| dx. \tag{3.16}$$

Due to (1.5), (1.21) and Taylor's expansion, we obtain

$$\tilde{p} - p_+ = \bar{\theta}_1 \theta_1 + \bar{\theta}_3 \theta_3 - \frac{\gamma-1}{2\tilde{v}} \bar{u}^2, \tag{3.17}$$

and

$$p - \tilde{p} + \frac{\tilde{p}}{\tilde{v}} \Phi_x - \frac{R}{\tilde{v}} (\theta - \tilde{\theta}) = O(1) (\phi_x^2 + W_x^2 + z^2 + Y^2). \tag{3.18}$$

Then, we have

$$I_3 \leq C\bar{\delta}(1+t)^{-1}\|\Psi\|^2 + C\bar{\delta}\|\Phi_x\|^2 + C\varepsilon_0(\|\phi_x\|^2 + \|W_x\|^2 + \|z\|^2 + \|\Psi_x\|^2), \tag{3.19}$$

where we have used the fact that

$$\begin{aligned} \int_{\mathbb{R}} Y^2|\Psi|dx &\leq C \int_{\mathbb{R}} |\Psi_x|^4|\Psi|dx + C \int_{\mathbb{R}} |\tilde{u}_x|^2|\Psi|^3dx \\ &\leq C\varepsilon_0\|\Psi_x\|^2 + C\bar{\delta}(1+t)^{-2}\|\Psi\|^2. \end{aligned} \tag{3.20}$$

For I_4 , we first have

$$\begin{aligned} \tilde{p} - \bar{p} &= -\frac{\bar{p}}{\bar{v}}(\tilde{v} - \bar{v}) + \frac{R}{\bar{v}}(\tilde{\theta} - \bar{\theta}) + O(1)[(\tilde{v} - \bar{v})^2 + (\tilde{\theta} - \bar{\theta})^2] \\ &= \frac{\gamma p_-}{v_-}\bar{\theta}_1\theta_1 + \frac{\gamma p_+}{v_+}\bar{\theta}_3\theta_3 \\ &\quad + O(\delta + \bar{\theta}_1^2 + \bar{\theta}_3^2)\frac{1}{1+t} \left(e^{-\frac{cx^2}{1+t}} + e^{-\frac{c(x-\lambda_1^-(1+t))^2}{1+t}} + e^{-\frac{c(x-\lambda_3^+(1+t))^2}{1+t}} \right), \end{aligned} \tag{3.21}$$

for some positive constant c . By a direct calculation, similar estimate also holds for $\tilde{p}\tilde{u} - \bar{p}\bar{u}$. Therefore,

$$\tilde{R}_i = O(\delta + \bar{\theta}_1^2 + \bar{\theta}_3^2)\frac{1}{1+t} \left(e^{-\frac{cx^2}{1+t}} + e^{-\frac{c(x-\lambda_1^-(1+t))^2}{1+t}} + e^{-\frac{c(x-\lambda_3^+(1+t))^2}{1+t}} \right), \tag{3.22}$$

for $i = 1, 2, 3$. By using (3.22), we get

$$I_4 \leq C\bar{\delta}(1+t)^{-1}\|\Psi\|^2 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.23}$$

By a direct calculation, one have

$$I_5 \leq \frac{1}{4}(1+t)^{-1}\|\Psi\|^2 + \lambda^2(\gamma - 1)^2(1+t)\|z\|^2. \tag{3.24}$$

Putting (3.14), (3.15), (3.19), (3.23) and (3.24) into (3.13), we get

$$\begin{aligned} \left| \int_{\mathbb{R}} \tilde{v}G_1\Psi dx \right| &\leq (C\bar{\delta} + \frac{1}{4})(1+t)^{-1}\|\Psi\|^2 + \lambda^2(\gamma - 1)^2(1+t)\|z\|^2 \\ &\quad + C(\bar{\delta} + \varepsilon_0)(\|\Phi_x\|^2 + \|z\|^2 + \|\Psi_x\|^2 + \|W_x\|^2) \\ &\quad + C\bar{\delta}(1+t)^{-\frac{1}{2}} + C\varepsilon_0\|\psi_x\|^2. \end{aligned} \tag{3.25}$$

Similarly, we get

$$\begin{aligned} \left| \int_{\mathbb{R}} G_2W dx \right| &\leq \int_{\mathbb{R}} \frac{R\kappa}{4p_+\tilde{v}}W_x^2 dx + C\bar{\delta}(1+t)^{-1}(\|\Psi\|^2 + \|W\|^2) \\ &\quad + C(\bar{\delta} + \varepsilon_0)(\|\Phi_x\|^2 + \|z\|^2 + \|\Psi_x\|^2 + \|W_x\|^2) \\ &\quad + C\bar{\delta}(1+t)^{-\frac{1}{2}} + C(\bar{\delta} + \varepsilon_0)\|(\phi, \psi, \zeta)_x\|^2. \end{aligned} \tag{3.26}$$

Substituting (3.10)-(3.12) and (3.25)-(3.26) into (3.9) and using the smallness of $\bar{\delta}$ and ε_0 , we get (3.7). \square

In the following, we show the estimates on Ψ_x .

Lemma 3.2. *It holds that*

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\mathbb{R}} \frac{\mu}{2\tilde{v}}\Phi_x^2 - \Phi_x\Psi dx \right) + \int_{\mathbb{R}} \frac{p_+}{4\tilde{v}}\Phi_x^2 dx \\ &\leq C_1 \int_{\mathbb{R}} \left(\mu\Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}}W_x^2 \right) dx + C_1\bar{\delta}(1+t)^{-\frac{3}{2}} + C_1\|z\|^2 + C_1\varepsilon_0\|\psi_x\|^2, \end{aligned} \tag{3.27}$$

for some positive constant C_1 .

Proof. From (2.4)₁ and (2.4)₂, it is obvious that

$$\frac{\mu}{\tilde{v}}\Phi_{xt} - \Psi_t + \frac{p_+}{\tilde{v}}\Phi_x = \frac{R}{\tilde{v}}W_x - G_1 - \frac{\mu}{\tilde{v}}\tilde{R}_{1x}. \tag{3.28}$$

Multiplying (3.28) by Φ_x , we obtain

$$\left(\frac{\mu}{2\tilde{v}}\Phi_x^2\right)_t - \left(\frac{\mu}{2\tilde{v}}\right)_t \Phi_x^2 - \Phi_x\Psi_t + \frac{p_+}{\tilde{v}}\Phi_x^2 = \left(\frac{R}{\tilde{v}}W_x - G_1 - \frac{\mu}{\tilde{v}}\tilde{R}_{1x}\right)\Phi_x. \tag{3.29}$$

Since

$$\Phi_x\Psi_t = (\Phi_x\Psi)_t - (\Phi_t\Psi)_x + \Psi_x^2 - \tilde{R}_1\Psi_x. \tag{3.30}$$

Putting (3.30) into (3.29) and integrating the resultant with respect to x over \mathbb{R} yield

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}} \frac{\mu}{2\tilde{v}} \Phi_x^2 - \Phi_x\Psi dx \right) + \int_{\mathbb{R}} \frac{p_+}{\tilde{v}} \Phi_x^2 dx \\ & \leq \int_{\mathbb{R}} \left(\frac{\mu}{2\tilde{v}} \right)_t \Phi_x^2 dx + \int_{\mathbb{R}} \Psi_x^2 dx - \int_{\mathbb{R}} \tilde{R}_1\Psi_x dx \\ & \quad + \int_{\mathbb{R}} \left(\frac{R}{\tilde{v}}W_x - G_1 - \frac{\mu}{\tilde{v}}\tilde{R}_{1x} \right) \Phi_x dx. \end{aligned} \tag{3.31}$$

By using (1.8), (1.18), (3.22) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}} \frac{\mu}{\tilde{v}} \Phi_x^2 - \Phi_x\Psi dx \right) + \int_{\mathbb{R}} \frac{p_+}{\tilde{v}} \Phi_x^2 dx \\ & \leq \left(C\bar{\delta} + \frac{1}{4} \right) \int_{\mathbb{R}} \frac{p_+}{\tilde{v}} \Phi_x^2 dx + C \int_{\mathbb{R}} \left(\mu\Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}} W_x^2 \right) dx \\ & \quad + C \int_{\mathbb{R}} G_1^2 dx + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{3.32}$$

Similar to the estimate (3.25) in the proof of Lemma 3.1, we have

$$\int_{\mathbb{R}} G_1^2 dx \leq C\varepsilon_0(\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2) + C\|z\|^2 + C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\varepsilon_0\|\psi_x\|^2. \tag{3.33}$$

By substituting (3.33) into (3.32) and using the smallness of $\bar{\delta}$ and ε_0 , we get (3.27). □

Choosing $\tilde{C}_1 > \max\{1, \frac{4}{\mu}, 4C_1\}$ and satisfying

$$\begin{aligned} & \tilde{C}_1 \int_{\mathbb{R}} \left(\frac{p_+}{2}\Phi^2 + \frac{\tilde{v}}{2}\Psi^2 + \frac{R^2}{2(\gamma-1)p_+}W^2 \right) dx + \int_{\mathbb{R}} \left(\frac{\mu}{2\tilde{v}}\Phi_x^2 - \Phi_x\Psi \right) dx \\ & \geq \frac{1}{2}\tilde{C}_1 \int_{\mathbb{R}} \left(\frac{p_+}{2}\Phi^2 + \frac{\tilde{v}}{2}\Psi^2 + \frac{R^2}{2(\gamma-1)p_+}W^2 \right) dx + \int_{\mathbb{R}} \frac{\mu}{4\tilde{v}}\Phi_x^2 dx, \end{aligned} \tag{3.34}$$

and

$$\frac{\tilde{C}_1}{2} - C_1 > \frac{\tilde{C}_1}{4}. \tag{3.35}$$

Denote

$$\mathcal{E}_1 = \tilde{C}_1 \int_{\mathbb{R}} \left(\frac{p_+}{2}\Phi^2 + \frac{\tilde{v}}{2}\Psi^2 + \frac{R^2}{2(\gamma-1)p_+}W^2 \right) dx + \int_{\mathbb{R}} \left(\frac{\mu}{2\tilde{v}}\Phi_x^2 - \Phi_x\Psi \right) dx, \tag{3.36}$$

and

$$\mathcal{D}_1 = \frac{\tilde{C}_1}{4} \int_{\mathbb{R}} \left(\mu\Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}}W_x^2 \right) dx + \int_{\mathbb{R}} \frac{p_+}{8\tilde{v}}\Phi_x^2 dx, \tag{3.37}$$

then it follows from Lemmas 3.1-3.2 and (3.34) that

$$\begin{aligned} \mathcal{E}_{1t} + \mathcal{D}_1 & \leq \left(C\bar{\delta} + \frac{1}{4} \right) (1+t)^{-1} \mathcal{E}_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}} \\ & \quad + C(\bar{\delta} + \varepsilon_0)\|(\phi, \psi, \zeta)_x\|^2 + C(1+t)\|z\|^2. \end{aligned} \tag{3.38}$$

3.2 Estimates on (ϕ, ψ, ζ, z)

Now we estimate the derivatives of (Φ, Ψ, W) . First of all, we denote

$$\vartheta = \theta + \frac{\lambda(\gamma - 1)}{R}z, \quad \eta = \vartheta - \tilde{\theta}. \tag{3.39}$$

Then the system (1.1) can be rewritten as

$$\begin{cases} \phi_t - \psi_x = -\tilde{R}_{1x}, \\ \psi_t + (p - \tilde{p})_x = \left(\frac{\mu}{v}u_x - \frac{\mu}{\tilde{v}}\tilde{u}_x\right)_x - \tilde{R}_{2x}, \\ \frac{R}{\gamma-1}\eta_t + pu_x - \tilde{p}\tilde{u}_x = \left(\frac{\kappa}{v}\theta_x - \frac{\kappa}{\tilde{v}}\tilde{\theta}_x\right)_x + \lambda\left(\frac{dz_x}{v^2}\right)_x + G_3, \\ z_t = \left(\frac{dz_x}{v}\right)_x - \varphi(\theta)z, \end{cases} \tag{3.40}$$

where

$$G_3 = \frac{\mu}{v}u_x^2 - \left(\frac{\mu}{\tilde{v}}\tilde{u}\tilde{u}_x\right)_x - \tilde{R}_{3x} + \frac{1}{2}(\tilde{u}^2)_t + \tilde{p}_x\tilde{u}. \tag{3.41}$$

Lemma 3.3. *It holds that*

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left[\frac{1}{2}\psi^2 + R\tilde{\theta}\hat{\Phi}\left(\frac{v}{\tilde{v}}\right) + \frac{R}{\gamma-1}\tilde{\theta}\hat{\Phi}\left(\frac{\vartheta}{\tilde{\theta}}\right) + \frac{\gamma_0}{2}z^2 \right] dx \\ & + \int_{\mathbb{R}} \left(\frac{\mu}{v}\psi_x^2 + \frac{\kappa}{v\vartheta}\eta_x^2 + \frac{\gamma_0}{4}\varphi(\theta)z^2 + \frac{\gamma_0}{4}\frac{d}{v}z_x^2 \right) dx \\ & \leq C\bar{\delta}(1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2) + C\bar{\delta}(1+t)^{-\frac{3}{2}}, \end{aligned} \tag{3.42}$$

where we denote the positive constant γ_0 satisfying

$$\gamma_0 \geq \max \left\{ \frac{2\lambda^2(\gamma - 1)}{\mu v \varphi(\theta)} \left(\frac{1}{\mu} + \frac{\kappa}{R^2\vartheta} \right), \frac{2\lambda^2 d}{\kappa \vartheta v^2} \right\}. \tag{3.43}$$

Proof. Set

$$\hat{\Phi}(s) = s - 1 - \ln s. \tag{3.44}$$

It is clear that $\hat{\Phi}'(1) = 0$ and $\hat{\Phi}(s)$ is strictly convex around $s = 1$. By a direct calculation, we obtain

$$\begin{aligned} \left[R\tilde{\theta}\hat{\Phi}\left(\frac{v}{\tilde{v}}\right) \right]_t &= R\tilde{\theta}_t\hat{\Phi}\left(\frac{v}{\tilde{v}}\right) + R\tilde{\theta}\left(-\frac{1}{v} + \frac{1}{\tilde{v}}\right)\phi_t \\ &+ R\tilde{\theta}\left(-\frac{v}{\tilde{v}^2} + \frac{1}{\tilde{v}}\right)\tilde{v}_t + R\tilde{\theta}\left(-\frac{1}{v} + \frac{1}{\tilde{v}}\right)\tilde{v}_t \\ &= R\tilde{\theta}\left(-\frac{1}{v} + \frac{1}{\tilde{v}}\right)\phi_t - \tilde{p}\hat{\Phi}\left(\frac{\tilde{v}}{v}\right)\tilde{v}_t + \tilde{v}\tilde{p}_t\hat{\Phi}\left(\frac{v}{\tilde{v}}\right), \end{aligned} \tag{3.45}$$

Multiplying (3.40)₂ by ψ , we get

$$\left(\frac{1}{2}\psi^2\right)_t - (p - \tilde{p})\psi_x + \left(\frac{\mu}{v}u_x - \frac{\mu}{\tilde{v}}\tilde{u}_x\right)\psi_x = -\tilde{R}_{2x}\psi + (\dots)_x. \tag{3.46}$$

Noticing that

$$p - \tilde{p} = R\tilde{\theta}\left(\frac{1}{v} - \frac{1}{\tilde{v}}\right) + \frac{R\eta}{v} - \frac{\lambda(\gamma - 1)}{v}z, \tag{3.47}$$

from (3.46) and (3.41)₁, we have

$$\begin{aligned} & \left(\frac{1}{2}\psi^2\right)_t - R\tilde{\theta}\left(\frac{1}{v} - \frac{1}{\tilde{v}}\right)\phi_t - \frac{R}{v}\eta\psi_x + \frac{\mu}{v}\psi_x^2 + \left(\frac{\mu}{v} - \frac{\mu}{\tilde{v}}\right)\tilde{u}_x\psi_x \\ & = -\tilde{R}_{2x}\psi - \frac{\lambda(\gamma - 1)}{v}z\psi_x + R\tilde{\theta}\left(\frac{1}{v} - \frac{1}{\tilde{v}}\right)\tilde{R}_{1x} + (\dots)_x. \end{aligned} \tag{3.48}$$

It is obvious that

$$\left[\tilde{\theta} \hat{\Phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) \right]_t = \left(1 - \frac{\tilde{\theta}}{\vartheta} \right) \eta_t - \hat{\Phi} \left(\frac{\tilde{\theta}}{\vartheta} \right) \tilde{\theta}_t. \tag{3.49}$$

By a direct calculation, we get

$$\begin{aligned} & \frac{R}{\gamma-1} \left(1 - \frac{\tilde{\theta}}{\vartheta} \right) \eta_t \\ &= \left(1 - \frac{\tilde{\theta}}{\vartheta} \right) \left\{ -p u_x + \tilde{p} \tilde{u}_x + \left(\frac{\kappa \theta_x}{v} - \frac{\kappa \tilde{\theta}}{\tilde{v}} \right)_x + \lambda \left(\frac{dz_x}{v^2} \right)_x + G_3 \right\} \\ &= -\frac{R\theta}{v\vartheta} \eta \psi_x + \frac{\eta}{\vartheta} (\tilde{p} - p) \tilde{u}_x - \left(\frac{\eta}{\vartheta} \right)_x \left(\frac{\kappa \theta_x}{v} - \frac{\kappa \tilde{\theta}_x}{\tilde{v}} \right) \\ &\quad - \left(\frac{\eta}{\vartheta} \right)_x \frac{\lambda dz_x}{v^2} + \frac{\eta}{\vartheta} G_3 + (\dots)_x \\ &= -\frac{R}{v} \eta \psi_x + \frac{\lambda(\gamma-1)}{v\vartheta} z \eta \psi_x + \frac{\eta}{\vartheta} (\tilde{p} - p) \tilde{u}_x - \frac{\kappa}{v\vartheta} \eta_x^2 + \frac{\kappa \lambda (\gamma-1)}{Rv\vartheta} z \eta_x \\ &\quad + \frac{\kappa \eta_x \tilde{\Phi}_x}{v\tilde{v}\vartheta} \tilde{\theta}_x + \frac{\eta \vartheta_x}{\vartheta^2} \left(\frac{\kappa \theta_x}{v} - \frac{\kappa \tilde{\theta}_x}{\tilde{v}} \right) - \left(\frac{\eta}{\vartheta} \right)_x \frac{\lambda dz_x}{v^2} + \frac{\eta}{\vartheta} G_3 + (\dots)_x. \end{aligned} \tag{3.50}$$

Adding (3.45) and (3.50) with (3.48) and integrating the resultant with respect to x over \mathbb{R} , we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left(R\tilde{\theta} \hat{\Phi} \left(\frac{v}{\tilde{v}} \right) + \frac{1}{2} \psi^2 + \frac{R}{\gamma-1} \tilde{\theta} \hat{\Phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) \right) dx + \int_{\mathbb{R}} \left(\frac{\mu}{v} \psi_x^2 + \frac{\kappa}{v\vartheta} \eta_x^2 \right) dx \\ &= - \int_{\mathbb{R}} \left(\tilde{p} \hat{\Phi} \left(\frac{\tilde{v}}{v} \right) \tilde{v}_t - \tilde{v} \tilde{p}_t \hat{\Phi} \left(\frac{v}{\tilde{v}} \right) + \frac{R}{\gamma-1} \hat{\Phi} \left(\frac{\tilde{\theta}}{\vartheta} \right) \tilde{\theta}_t \right) dx \\ &\quad - \int_{\mathbb{R}} \left(\frac{\mu}{v} - \frac{\mu}{\tilde{v}} \right) \tilde{u}_x \psi_x dx - \int_{\mathbb{R}} \tilde{R}_{2x} \psi dx + \int_{\mathbb{R}} R\tilde{\theta} \left(\frac{1}{v} - \frac{1}{\tilde{v}} \right) \tilde{R}_{1x} dx \\ &\quad - \int_{\mathbb{R}} \frac{\lambda(\gamma-1)}{v} z \psi_x dx + \int_{\mathbb{R}} \frac{\lambda(\gamma-1)}{v\vartheta} z \eta \psi_x dx - \int_{\mathbb{R}} \frac{\kappa \lambda (\gamma-1)}{Rv\vartheta} z \eta_x dx \\ &\quad + \int_{\mathbb{R}} \frac{\eta}{\vartheta} (\tilde{p} - p) \tilde{u}_x dx + \int_{\mathbb{R}} \frac{\kappa \eta_x \tilde{\Phi}_x}{v\tilde{v}\vartheta} \tilde{\theta}_x dx + \int_{\mathbb{R}} \frac{\eta \vartheta_x}{\vartheta^2} \left(\frac{\kappa \theta_x}{v} - \frac{\kappa \tilde{\theta}_x}{\tilde{v}} \right) dx \\ &\quad - \int_{\mathbb{R}} \left(\frac{\eta}{\vartheta} \right)_x \frac{\lambda dz_x}{v^2} dx + \int_{\mathbb{R}} \frac{\eta}{\vartheta} G_3 dx. \end{aligned} \tag{3.51}$$

Now we will estimate the right hand side terms of (3.51). Noting that $\hat{\Phi}(s)$ is strictly convex around $s = 1$. There exist positive constants c_1 and c_2 such that

$$c_1 \phi^2 \leq \hat{\Phi} \left(\frac{v}{\tilde{v}} \right) \leq c_2 \phi^2, \tag{3.52}$$

and

$$c_1 \eta^2 \leq \hat{\Phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) \leq c_2 \eta^2. \tag{3.53}$$

Therefore, by using (1.8), (1.18) and (2.3), we have

$$\int_{\mathbb{R}} \left| \tilde{p} \hat{\Phi} \left(\frac{\tilde{v}}{v} \right) \tilde{v}_t \right| dx + \int_{\mathbb{R}} \left| \tilde{v} \tilde{p}_t \hat{\Phi} \left(\frac{v}{\tilde{v}} \right) \right| dx \leq C \bar{\delta} (1+t)^{-1} \|\Psi_x\|^2, \tag{3.54}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \left| \hat{\Phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) \tilde{\theta}_t \right| dx &\leq C \bar{\delta} (1+t)^{-1} \|\eta\|^2 \\ &\leq C \bar{\delta} (1+t)^{-1} (\|W_x\|^2 + \|Y\|^2) \\ &\leq C \bar{\delta} (1+t)^{-1} (\|W_x\|^2 + \|\Psi_x\|^2) + C \bar{\delta} \varepsilon_0 (1+t)^{-\frac{5}{2}}. \end{aligned} \tag{3.55}$$

By the Cauchy inequality, (1.8) and (1.18), we get

$$\int_{\mathbb{R}} \left| \left(\frac{\mu}{v} - \frac{\mu}{\tilde{v}} \right) \tilde{u}_x \psi_x \right| dx = \int_{\mathbb{R}} \left| \frac{\mu \Phi_x}{v \tilde{v}} \tilde{u}_x \psi_x \right| dx \leq C \bar{\delta} \|\psi_x\|^2 + C \bar{\delta} (1+t)^{-2} \|\Phi_x\|^2, \tag{3.56}$$

and

$$\int_{\mathbb{R}} \left| \frac{\kappa \eta_x \Phi_x}{v \tilde{v} \vartheta} \tilde{\theta}_x \right| dx \leq C \bar{\delta} \|\eta_x\|^2 + C \bar{\delta} (1+t)^{-1} \|\Phi_x\|^2. \tag{3.57}$$

By using (3.22) and the Cauchy inequality, we have

$$\int_{\mathbb{R}} |\tilde{R}_{2x} \psi| dx \leq C \bar{\delta} (1+t)^{-1} \|\Psi_x\|^2 + C \bar{\delta} (1+t)^{-\frac{3}{2}}, \tag{3.58}$$

and

$$\int_{\mathbb{R}} \left| R \tilde{\theta} \left(\frac{1}{v} - \frac{1}{\tilde{v}} \right) \tilde{R}_{1x} \right| dx \leq C \bar{\delta} (1+t)^{-1} \|\Phi_x\|^2 + C \bar{\delta} (1+t)^{-\frac{3}{2}}. \tag{3.59}$$

It follows from the Cauchy inequality and the assumption (3.10) that

$$\begin{aligned} & \int_{\mathbb{R}} \left| \frac{\lambda(\gamma-1)}{v} z \psi_x \right| + \left| \frac{\kappa \lambda(\gamma-1)}{R v \vartheta} z \eta_x \right| dx \\ & \leq \int_{\mathbb{R}} \frac{\mu}{4v} \psi_x^2 dx + \int_{\mathbb{R}} \frac{\kappa}{4v \vartheta} \eta_x^2 dx + \int_{\mathbb{R}} \frac{\lambda^2(\gamma-1)^2}{\mu v} z^2 dx + \int_{\mathbb{R}} \frac{\kappa \lambda^2(\gamma-1)^2}{R^2 v \vartheta} z^2 dx, \end{aligned} \tag{3.60}$$

and

$$\int_{\mathbb{R}} \frac{\lambda(\gamma-1)}{v \vartheta} z \eta \psi_x dx \leq C \varepsilon_0 (\|z\|^2 + \|\psi_x\|^2). \tag{3.61}$$

By using (3.47) and the Cauchy inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}} \left| \frac{\eta}{\theta} (\tilde{p} - p) \tilde{u}_x \right| dx \\ & \leq C \bar{\delta} (1+t)^{-1} (\|\Phi_x\|^2 + \|\eta\|^2 + \|z\|^2) \\ & \leq C \bar{\delta} (1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2 + \|z\|^2) + C \bar{\delta} (1+t)^{-\frac{5}{2}}. \end{aligned} \tag{3.62}$$

Similarly, we have

$$\begin{aligned} & \int_{\mathbb{R}} \left| \frac{\eta \vartheta_x}{\vartheta^2} \left(\frac{\kappa \theta_x}{v} - \frac{\kappa \tilde{\theta}_x}{\tilde{v}} \right) \right| dx \leq C (\bar{\delta} + \varepsilon_0) (\|\eta_x\|^2 + \|\zeta_x\|^2) \\ & \quad + C \bar{\delta} (1+t)^{-1} (\|\phi\|^2 + \|\eta\|^2) \\ & \leq C (\bar{\delta} + \varepsilon_0) (\|\eta_x\|^2 + \|z_x\|^2) + C \bar{\delta} (1+t)^{-\frac{5}{2}} \\ & \quad + C \bar{\delta} (1+t)^{-1} (\|W_x\|^2 + \|\Psi_x\|^2 + \|\Phi_x\|^2), \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} & \int_{\mathbb{R}} \left| \left(\frac{\eta}{\vartheta} \right)_x \frac{\lambda dz_x}{v^2} \right| dx \leq \int_{\mathbb{R}} \frac{\kappa}{4v \vartheta} \eta_x^2 dx + \int_{\mathbb{R}} \frac{\lambda^2 d^2}{\kappa \vartheta v^3} z_x^2 dx \\ & \quad + C \bar{\delta} (1+t)^{-1} (\|\Psi_x\|^2 + \|W_x\|^2) \\ & \quad + C (\varepsilon_0 + \bar{\delta}) (\|z_x\|^2 + \|\eta_x\|^2) + C \bar{\delta} (1+t)^{-\frac{5}{2}}. \end{aligned} \tag{3.64}$$

By the Cauchy inequality, we have

$$\int_{\mathbb{R}} \left| \frac{\eta}{\theta} G_3 \right| dx \leq C \varepsilon_0 \|\psi_x\|^2 + C \bar{\delta} (1+t)^{-1} (\|\Psi_x\|^2 + \|W_x\|^2) + C \bar{\delta} (1+t)^{-\frac{3}{2}}. \tag{3.65}$$

If we plug (3.54)-(3.65) back into (3.51), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left(R\tilde{\theta}\hat{\Phi} \left(\frac{v}{\tilde{v}} \right) + \frac{1}{2}\psi^2 + \frac{R}{\gamma-1}\tilde{\theta}\hat{\Phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) \right) dx + \int_{\mathbb{R}} \left(\frac{\mu}{2v}\psi_x^2 + \frac{\kappa}{4v\vartheta}\eta_x^2 \right) dx \\ \leq & \int_{\mathbb{R}} \frac{\lambda^2(\gamma-1)^2}{\mu v} \left(\frac{1}{\mu} + \frac{\kappa}{R^2\vartheta} \right) z^2 dx + \int_{\mathbb{R}} \frac{\lambda^2 d^2}{\kappa\vartheta v^3} z_x^2 dx + C\bar{\delta}(1+t)^{-\frac{3}{2}} \\ & + C\bar{\delta}(1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2) + C(\varepsilon_0 + \bar{\delta}) (\|z\|^2 + \|z_x\|^2), \end{aligned} \tag{3.66}$$

where we have used the fact that the constants $\bar{\delta}$ and ε_0 are small.

Multiplying (3.40)₄ by z and integrating the resultant with respect to x over \mathbb{R} , we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} z^2 dx + \int_{\mathbb{R}} \varphi(\theta)z^2 dx + \int_{\mathbb{R}} \frac{d}{v} z_x^2 dx = 0. \tag{3.67}$$

From (3.66) + γ_0 (3.67) and the smallness of $\bar{\delta}$ and ε_0 , we get (3.42). □

Now we show the $L_t^2 L_x^2$ -norm estimate of ϕ_x as follows.

Lemma 3.4. *It holds that*

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left(\frac{\mu}{2\tilde{v}}\phi_x^2 - \phi_x\psi \right) dx + \int_{\mathbb{R}} \frac{\tilde{p}}{2\tilde{v}}\phi_x^2 dx \\ \leq & C_2 \int_{\mathbb{R}} \left(\frac{\mu}{v}\psi_x^2 + \frac{\kappa}{v\vartheta}\eta_x^2 + \frac{\gamma_0}{4} \frac{d}{v} z_x^2 \right) dx \\ & + C_2\bar{\delta}(1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2 + \|z\|^2) \\ & + C_2\bar{\delta}(1+t)^{-\frac{5}{2}} + C_2\varepsilon_0\|\psi_{xx}\|^2, \end{aligned} \tag{3.68}$$

for some positive constant C_2 .

Proof. From (3.40)₁ and (3.40)₂, we have

$$\frac{\mu}{\tilde{v}}\phi_{xt} - \psi_t - (p - \tilde{p})_x = - \left(\frac{\mu}{\tilde{v}} \right)_x \psi_x - \left(\frac{\mu}{v} - \frac{\mu}{\tilde{v}} \right) u_x + \tilde{R}_{2x} - \frac{\mu}{\tilde{v}}\tilde{R}_{1xx}. \tag{3.69}$$

Multiplying (3.69) by ϕ_x , we obtain

$$\begin{aligned} & \left(\frac{\mu}{2\tilde{v}}\phi_x^2 \right) - \left(\frac{\mu}{2\tilde{v}} \right)_t \phi_x^2 - \psi_t\phi_x - (p - \tilde{p})_x\phi_x \\ = & \left[- \left(\frac{\mu}{\tilde{v}} \right)_x \psi_x + \left(\frac{\mu\Phi_x}{v\tilde{v}} u_x \right)_x + \tilde{R}_{2x} - \frac{\mu}{\tilde{v}}\tilde{R}_{1xx} \right] \phi_x. \end{aligned} \tag{3.70}$$

Note that

$$\begin{aligned} -(p - \tilde{p})_x &= \frac{\tilde{p}}{\tilde{v}}\phi_x - \frac{R}{\tilde{v}}\eta_x + \frac{\lambda(\gamma-1)}{\tilde{v}}z_x + \left(\frac{p}{v} - \frac{\tilde{p}}{\tilde{v}} \right) v_x - \left(\frac{R}{v} - \frac{R}{\tilde{v}} \right) \theta_x \\ &= \frac{\tilde{p}}{\tilde{v}}\phi_x - \frac{R}{\tilde{v}}\eta_x + \frac{\lambda(\gamma-1)}{\tilde{v}}z_x - \left(\frac{R}{v} - \frac{R}{\tilde{v}} \right) \theta_x \\ &+ \left[\frac{p}{v^2}(W_x - Y) - \frac{\lambda(\gamma-1)}{v^2}z + \left(\frac{R}{v^2} - \frac{R}{\tilde{v}^2} \right) \tilde{\theta} \right] v_x, \end{aligned} \tag{3.71}$$

and

$$\phi_x\psi_t = (\phi_x\psi)_t - (\phi_t\psi)_x + \psi_x^2 - \tilde{R}_{1x}\psi_x. \tag{3.72}$$

Putting (3.71) and (3.72) into (3.70) and integrating the resultant with respect to x over \mathbb{R} , we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left(\frac{\mu}{2\tilde{v}} \phi_x^2 - \phi_x \psi \right) dx + \int_{\mathbb{R}} \frac{\tilde{p}}{\tilde{v}} \phi_x^2 dx \\ &= \int_{\mathbb{R}} \left(\frac{\mu}{2\tilde{v}} \right)_t \phi_x^2 dx + \int_{\mathbb{R}} \psi_x^2 dx - \int_{\mathbb{R}} \tilde{R}_{1x} \psi_x dx + \int_{\mathbb{R}} \frac{R}{\tilde{v}} \eta_x \phi_x dx \\ & \quad - \int_{\mathbb{R}} \frac{\lambda(\gamma-1)}{\tilde{v}} z_x \phi_x dx + \int_{\mathbb{R}} \left(\frac{R}{\tilde{v}} - \frac{R}{v} \right) \theta_x \phi_x dx - \int_{\mathbb{R}} \left(\frac{p}{v} - \frac{\tilde{p}}{\tilde{v}} \right) v_x \phi_x dx \\ & \quad + \int_{\mathbb{R}} \left[- \left(\frac{\mu}{\tilde{v}} \right)_x \psi_x + \left(\frac{\mu \Phi_x}{v \tilde{v}} u_x \right)_x + \tilde{R}_{2x} - \frac{\mu}{\tilde{v}} \tilde{R}_{1xx} \right] \phi_x dx. \end{aligned} \tag{3.73}$$

Now we estimate the right hand side of (3.73) as follows. It is clear that

$$\int_{\mathbb{R}} \left(\left| \left(\frac{\mu}{2\tilde{v}} \right)_t \phi_x^2 \right| + \psi_x^2 + \left| \left(\frac{\mu}{\tilde{v}} \right)_x \psi_x^2 \right| \right) dx \leq \int_{\mathbb{R}} \frac{\tilde{p}}{8\tilde{v}} \phi_x^2 dx + C \int_{\mathbb{R}} \frac{\mu}{v} \psi_x^2 dx. \tag{3.74}$$

By using (3.22) and the Cauchy inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}} \left(|\tilde{R}_{1x} \psi_x| + |\tilde{R}_{2x} \phi_x| + \left| \frac{\mu}{\tilde{v}} \tilde{R}_{1xx} \right| \right) dx \\ & \leq C\bar{\delta} (\|\psi_x\|^2 + \|\phi_x\|^2) + C\bar{\delta}(1+t)^{-\frac{5}{2}}. \end{aligned} \tag{3.75}$$

It follows from the Cauchy inequality that

$$\begin{aligned} & \int_{\mathbb{R}} \left(\left| \frac{R}{\tilde{v}} \eta_x \phi_x \right| + \left| \frac{\lambda(\gamma-1)}{\tilde{v}} z_x \phi_x \right| \right) dx \\ & \leq \int_{\mathbb{R}} \frac{\tilde{p}}{8\tilde{v}} \phi_x^2 dx + C \int_{\mathbb{R}} \left(\frac{\kappa}{v\vartheta} \eta_x^2 + \frac{\gamma_0}{4} \frac{d}{v} z_x^2 \right) dx. \end{aligned} \tag{3.76}$$

By using the Cauchy inequality and (1.8) and (1.18), we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \left| \left(\frac{R}{v} - \frac{R}{\tilde{v}} \right) \theta_x \phi_x \right| dx \\ & \leq C\varepsilon_0 (\|\eta_x\|^2 + \|z_x\|^2) + C(\bar{\delta} + \varepsilon_0) \|\phi_x\|^2 + C\bar{\delta}(1+t)^{-1} \|\Phi_x\|^2. \end{aligned} \tag{3.77}$$

Similarly, we have

$$\begin{aligned} & \int_{\mathbb{R}} \left| \left(\frac{p}{v} - \frac{\tilde{p}}{\tilde{v}} \right) v_x \phi_x \right| dx \leq C(\bar{\delta} + \varepsilon_0) \|\phi_x\|^2 + C\bar{\delta}(1+t)^{-\frac{5}{2}} \\ & \quad + C\bar{\delta}(1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2 + \|z\|^2). \end{aligned} \tag{3.78}$$

By using the Cauchy inequality, the Hölder inequality, (1.8) and (1.18), we get

$$\begin{aligned} & \int_{\mathbb{R}} \left| \left(\frac{\mu \Phi_x}{v \tilde{v}} u_x \right)_x \phi_x \right| dx \\ & \leq C \int_{\mathbb{R}} \phi_x^2 |\psi_x| dx + C \int_{\mathbb{R}} \phi_x^2 |\tilde{u}_x| dx + C \int_{\mathbb{R}} |\Phi_x \phi_x^2 \psi_x| dx \\ & \quad + C \int_{\mathbb{R}} |\Phi_x \tilde{v}_x \phi_x \psi_x| dx + C \int_{\mathbb{R}} |\Phi_x \tilde{u}_x \phi_x^2| dx + C \int_{\mathbb{R}} |\Phi_x \tilde{v}_x \tilde{u}_x \phi_x| dx \\ & \quad + C \int_{\mathbb{R}} |\Phi_x \psi_{xx} \phi_x| dx + C \int_{\mathbb{R}} |\Phi_x \tilde{u}_{xx} \phi_x| dx \\ & \leq C(\bar{\delta} + \varepsilon_0) \|\phi_x\|^2 + C\bar{\delta}(1+t)^{-1} \|\Phi_x\|^2 + C\varepsilon_0 \|\psi_{xx}\|^2 \\ & \quad + C\bar{\delta} \|\psi_x\|^2 + C \int_{\mathbb{R}} \phi_x^2 |\psi_x| dx \\ & \leq C(\bar{\delta} + \varepsilon_0) \|\phi_x\|^2 + C\bar{\delta}(1+t)^{-1} \|\Phi_x\|^2 + C\varepsilon_0 \|\psi_{xx}\|^2 + C\bar{\delta} \|\psi_x\|^2, \end{aligned} \tag{3.79}$$

where we have used the fact that

$$\begin{aligned} \int_{\mathbb{R}} \phi_x^2 |\psi_x| dx &\leq C \|\phi_x\|^2 \|\psi_x\|^{\frac{1}{2}} \|\psi_{xx}\|^{\frac{1}{2}} \\ &\leq C \|\phi_x\|^{\frac{1}{2}} \|\psi_x\|^{\frac{1}{2}} (\|\phi_x\|^2 + \|\psi_{xx}\|^2) \\ &\leq C \varepsilon_0 (\|\phi_x\|^2 + \|\psi_{xx}\|^2). \end{aligned} \tag{3.80}$$

Plugging (3.74)-(3.80) into (3.73) and using the smallness of ε_0 and $\bar{\delta}$, we get (3.68). □

3.3 Higher Order Estimates on (ψ, ζ, z)

In this subsection, we establish the $L_t^\infty L_x^2$ -norm estimates of the higher order derivatives of (ψ, η, z) and the $L_t^2 L_x^2$ -norm estimates of $(\psi_{xx}, \eta_{xx}, z_x, z_{xx})$ by using the energy method.

Lemma 3.5. *It holds that*

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{1}{2} \psi_x^2 + \frac{R}{2(\gamma-1)} \eta_x^2 + \frac{\gamma_1}{2} z_x^2 \right) dx \\ &+ \int_{\mathbb{R}} \frac{\mu}{2v} \psi_{xx}^2 dx + \int_{\mathbb{R}} \frac{\kappa}{2v} \eta_{xx}^2 dx + \int_{\mathbb{R}} \frac{\varphi(\theta)\gamma_1}{4} z_x^2 dx + \int_{\mathbb{R}} \frac{\lambda d \gamma_1}{4v^2} z_{xx}^2 dx \\ &\leq C_3 (\bar{\delta} + \varepsilon_0) \|\psi_x\|^2 + C_3 (\|z\|^2 + \|\phi_x\|^2) + C_3 \int_{\mathbb{R}} \left(\frac{\kappa}{v\vartheta} \eta_x^2 + \frac{\gamma_0}{4} \frac{d}{v} z_x^2 \right) dx \\ &+ C_3 \bar{\delta} (1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2 + \|z\|^2) + C_3 \bar{\delta} (1+t)^{-\frac{5}{2}}, \end{aligned} \tag{3.81}$$

where C_3 is a positive constant and γ_1 denotes a positive constant satisfying

$$\gamma_1 \geq \frac{4\lambda v}{d} \left(\frac{d^2}{\kappa v^2} + \frac{(\gamma-1)^2 \kappa}{R^2} \right). \tag{3.82}$$

Proof. Multiplying (3.40)₂ and (3.40)₃ by $-\psi_{xx}$ and $-\eta_{xx}$ respectively, and integrating the resultant with respect to x over \mathbb{R} , we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{1}{2} \psi_x^2 + \frac{R}{2(\gamma-1)} \eta_x^2 \right) dx + \int_{\mathbb{R}} \frac{\mu}{v} \psi_{xx}^2 dx + \int_{\mathbb{R}} \frac{\kappa}{v} \eta_{xx}^2 dx \\ &= \int_{\mathbb{R}} (p - \tilde{p})_x \psi_{xx} dx + \int_{\mathbb{R}} \frac{\mu v_x}{v^2} \psi_x \psi_{xx} dx + \int_{\mathbb{R}} \left(\frac{\mu \Phi_x}{v \tilde{v}} \tilde{u}_x \right)_x \psi_{xx} dx \\ &+ \int_{\mathbb{R}} \tilde{R}_{2x} \psi_{xx} dx + \int_{\mathbb{R}} (p u_x - \tilde{p} \tilde{u}_x) \eta_{xx} dx + \int_{\mathbb{R}} \frac{\kappa v_x}{v^2} \eta_x \eta_{xx} dx \\ &- \frac{\lambda(\gamma-1)}{R} \int_{\mathbb{R}} \frac{\kappa v_x}{v^2} z_x \eta_{xx} dx + \int_{\mathbb{R}} \left(\frac{\kappa \Phi_x}{v \tilde{v}} \tilde{\theta}_x \right)_x \eta_{xx} dx - \int_{\mathbb{R}} G_3 \eta_{xx} dx \\ &+ \lambda \int_{\mathbb{R}} \left(\frac{dz_x}{v^2} \right)_x \eta_{xx} dx + \frac{\lambda(\gamma-1)}{R} \int_{\mathbb{R}} \frac{\kappa}{v} z_{xx} \eta_{xx} dx. \end{aligned} \tag{3.83}$$

Now we will estimate the right hand side of (3.83). By using the Cauchy inequality, (1.8) and (1.18), we get

$$\int_{\mathbb{R}} \left(\frac{\mu \Phi_x}{v \tilde{v}} \tilde{u}_x \right)_x \psi_{xx} dx \leq C \bar{\delta} \|\psi_{xx}\|^2 + C \bar{\delta} \|\phi_x\|^2 + C \bar{\delta} (1+t)^{-1} \|\Phi_x\|^2. \tag{3.84}$$

Similarly, we have

$$\int_{\mathbb{R}} \left(\frac{\kappa \Phi_x}{v \tilde{v}} \tilde{\theta}_x \right)_x \eta_{xx} dx \leq C \bar{\delta} \|\eta_{xx}\|^2 + C \bar{\delta} \|\phi_x\|^2 + C \bar{\delta} (1+t)^{-1} \|\Phi_x\|^2. \tag{3.85}$$

By using (3.22) and the Cauchy inequality, we obtain

$$\int_{\mathbb{R}} \tilde{R}_{2x} \psi_{xx} dx \leq C\bar{\delta} \|\psi_{xx}\|^2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}. \quad (3.86)$$

By using the Cauchy inequality, (1.8), (1.18), (2.3) and (3.71), we have

$$\begin{aligned} & \int_{\mathbb{R}} |(p - \tilde{p})_x \psi_{xx}| dx \\ & \leq C(\|\eta_x\|^2 + \|z_x\|^2 + \|\phi_x\|^2) + \int_{\mathbb{R}} \frac{\mu}{8v} \psi_{xx}^2 dx \\ & \quad + C\bar{\delta}(1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2 + \|z\|^2) + C\bar{\delta}(1+t)^{-\frac{5}{2}}. \end{aligned} \quad (3.87)$$

By using the Cauchy inequality and the Hölder inequality, we get

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\mu v_x}{v^2} \psi_x \psi_{xx} \right| dx & \leq C\bar{\delta} (\|\psi_x\|^2 + \|\psi_{xx}\|^2) + C \int_{\mathbb{R}} |\phi_x| |\psi_x| |\psi_{xx}| dx \\ & \leq C(\bar{\delta} + \varepsilon_0) (\|\psi_x\|^2 + \|\psi_{xx}\|^2), \end{aligned} \quad (3.88)$$

where we have used the fact that

$$\int_{\mathbb{R}} |\phi_x| |\psi_x| |\psi_{xx}| dx \leq C \|\psi_x\|^{\frac{1}{2}} \|\psi_{xx}\|^{\frac{1}{2}} \|\phi_x\| \|\psi_{xx}\| \leq C\varepsilon_0 (\|\psi_x\|^2 + \|\psi_{xx}\|^2). \quad (3.89)$$

Similarly, we have

$$\int_{\mathbb{R}} \frac{\kappa v_x}{v^2} \eta_x \eta_{xx} dx \leq C(\bar{\delta} + \varepsilon_0) (\|\eta_x\|^2 + \|\eta_{xx}\|^2), \quad (3.90)$$

$$\frac{\lambda(\gamma-1)}{R} \int_{\mathbb{R}} \frac{\kappa v_x}{v^2} z_x \eta_{xx} dx \leq C(\bar{\delta} + \varepsilon_0) (\|z_x\|^2 + \|z_{xx}\|^2 + \|\eta_{xx}\|^2), \quad (3.91)$$

$$\begin{aligned} \lambda \int_{\mathbb{R}} \left(\frac{dz_x}{v^2} \right)_x \eta_{xx} dx & \leq C(\varepsilon_0 + \bar{\delta}) (\|z_x\|^2 + \|z_{xx}\|^2 + \|\eta_{xx}\|^2) \\ & \quad + \int_{\mathbb{R}} \frac{\kappa}{8v} \eta_{xx}^2 dx + \int_{\mathbb{R}} \frac{2\lambda^2 d^2}{\kappa v^3} z_{xx}^2 dx, \end{aligned} \quad (3.92)$$

and

$$\begin{aligned} \int_{\mathbb{R}} |G_3 \eta_{xx}| dx & \leq C \int_{\mathbb{R}} \psi_x^2 |\eta_{xx}| dx + C\bar{\delta} (\|\psi_x\|^2 + \|\eta_{xx}\|^2) + C\bar{\delta}(1+t)^{-\frac{5}{2}} \\ & \leq C(\bar{\delta} + \varepsilon_0) (\|\psi_x\|^2 + \|\eta_{xx}\|^2) + C\varepsilon_0 \|\psi_{xx}\|^2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}. \end{aligned} \quad (3.93)$$

Noting that

$$p u_x - \tilde{p} \tilde{u}_x = p \psi_x + \left[\frac{R}{v} (W_x - Y) - \frac{\lambda(\gamma-1)}{v} z - \frac{R\Phi_x}{v\tilde{v}} \tilde{\theta}_x \right] \tilde{u}_x,$$

and using the Cauchy inequality, (1.8), (1.18) and (2.3), we have

$$\begin{aligned} \int_{\mathbb{R}} |(p u_x - \tilde{p} \tilde{u}_x) \eta_{xx}| dx & \leq \int_{\mathbb{R}} \frac{\mu}{8v} \eta_{xx}^2 dx + C \|\psi_x\|^2 + C\bar{\delta}(1+t)^{-\frac{5}{2}} \\ & \quad + C\bar{\delta}(1+t)^{-1} (\|\Psi_x\|^2 + \|W_x\|^2 + \|z\|^2). \end{aligned} \quad (3.94)$$

It follows from the Cauchy inequality that

$$\frac{\lambda(\gamma-1)}{R} \int_{\mathbb{R}} \frac{\kappa}{v} z_{xx} \eta_{xx} dx \leq \int_{\mathbb{R}} \frac{\kappa}{8v} \eta_{xx}^2 dx + \int_{\mathbb{R}} \frac{2\lambda^2(\gamma-1)^2 \kappa}{R^2 v} z_{xx}^2 dx. \quad (3.95)$$

Combining (3.83) and (3.84)-(3.95), we see that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \left(\frac{1}{2} \psi_x^2 + \frac{R}{2(\gamma-1)} \eta_x^2 \right) dx + \int_{\mathbb{R}} \frac{\mu}{2v} \psi_{xx}^2 dx + \int_{\mathbb{R}} \frac{\kappa}{2v} \eta_{xx}^2 dx \\ & \leq \int_{\mathbb{R}} \frac{2\lambda^2}{v} \left(\frac{d^2}{\kappa v^2} + \frac{(\gamma-1)^2 \kappa}{R^2} \right) z_{xx}^2 dx + C(\bar{\delta} + \varepsilon_0) (\|\psi_x\|^2 + \|z_{xx}\|^2) \\ & \quad + C\bar{\delta}(1+t)^{-\frac{5}{2}} + C(\|\phi_x\|^2 + \|\eta_x\|^2 + \|z_x\|^2) \\ & \quad + C\bar{\delta}(1+t)^{-1} (\|\Phi_x\|^2 + \|\Psi_x\|^2 + \|W_x\|^2 + \|z\|^2). \end{aligned} \tag{3.96}$$

Multiplying (3.40)₄ by $-z_{xx}$, and integrating the resultant with respect to x over \mathbb{R} , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} z_x^2 dx + \int_{\mathbb{R}} \varphi(\theta) z_x^2 dx + \int_{\mathbb{R}} \frac{\lambda d}{v^2} z_{xx}^2 dx \\ & = - \int_{\mathbb{R}} \varphi(\theta)_x z z_x dx - \int_{\mathbb{R}} \left(\frac{d}{v} \right)_x z_x z_{xx} dx. \end{aligned} \tag{3.97}$$

By using the Cauchy inequality, (1.18) and the assumption (3.3), we have

$$\int_{\mathbb{R}} \varphi(\theta)_x z z_x dx \leq C(\varepsilon_0 + \bar{\delta}) (\|\eta_x\|^2 + \|z\|^2 + \|z_x\|^2), \tag{3.98}$$

and

$$\int_{\mathbb{R}} \left(\frac{d}{v} \right)_x z_x z_{xx} dx \leq C(\varepsilon_0 + \bar{\delta}) (\|z_x\|^2 + \|z_{xx}\|^2). \tag{3.99}$$

Putting (3.98) and (3.99) into (3.97) and using the smallness of $\bar{\delta}$ and ε_0 , we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} z_x^2 dx + \int_{\mathbb{R}} \frac{\varphi(\theta)}{2} z_x^2 dx + \int_{\mathbb{R}} \frac{\lambda d}{2v^2} z_{xx}^2 dx \leq C(\bar{\delta} + \varepsilon_0) (\|z\|^2 + \|\eta_x\|^2). \tag{3.100}$$

From (3.96) + $\gamma_1(3.100)$, by using the smallness of ε_0 and $\bar{\delta}$, we get (3.81). □

Choosing two positive constants \tilde{C}_2 and \tilde{C}_3 such that

$$\begin{aligned} & \tilde{C}_2 \int_{\mathbb{R}} \left[\frac{1}{2} \psi^2 + R\tilde{\theta}\tilde{\phi} \left(\frac{v}{\tilde{v}} \right) + \frac{R}{\gamma-1} \tilde{\theta}\tilde{\phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) + \frac{\gamma_0}{2} z^2 \right] dx + \tilde{C}_3 \int_{\mathbb{R}} \left(\frac{\mu}{2v} \phi_x^2 - \phi_x \psi \right) dx \\ & > \frac{1}{2} \tilde{C}_2 \int_{\mathbb{R}} \left[\frac{1}{2} \psi^2 + R\tilde{\theta}\tilde{\phi} \left(\frac{v}{\tilde{v}} \right) + \frac{R}{\gamma-1} \tilde{\theta}\tilde{\phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) + \frac{\gamma_0}{2} z^2 \right] dx + \frac{\tilde{C}_3}{4} \int_{\mathbb{R}} \frac{\mu}{v} \phi_x^2 dx, \end{aligned} \tag{3.101}$$

$$\tilde{C}_2 - \tilde{C}_3 C_2 - C_3 > \frac{1}{2} \tilde{C}_2, \tag{3.102}$$

$$\frac{\tilde{C}_2}{2} \int_{\mathbb{R}} \frac{\gamma_0}{4} \varphi(\theta) z^2 dx - C_3 \|z\|^2 > \frac{\tilde{C}_2}{2} \int_{\mathbb{R}} \frac{\gamma_0}{8} \varphi(\theta) z^2 dx, \tag{3.103}$$

and

$$\tilde{C}_3 \int_{\mathbb{R}} \frac{\tilde{p}}{2\tilde{v}} \phi_x^2 dx - C_3 \|\phi_x\|^2 > \tilde{C}_3 \int_{\mathbb{R}} \frac{\tilde{p}}{4\tilde{v}} \phi_x^2 dx. \tag{3.104}$$

Let

$$\begin{aligned} \mathcal{E}_2 = & \tilde{C}_2 \int_{\mathbb{R}} \left[\frac{1}{2} \psi^2 + R\tilde{\theta}\tilde{\phi} \left(\frac{v}{\tilde{v}} \right) + \frac{R}{\gamma-1} \tilde{\theta}\tilde{\phi} \left(\frac{\vartheta}{\tilde{\theta}} \right) + \frac{\gamma_0}{2} z^2 \right] dx \\ & + \tilde{C}_3 \int_{\mathbb{R}} \left(\frac{\mu}{2v} \phi_x^2 - \phi_x \psi \right) dx + \int_{\mathbb{R}} \left(\frac{1}{2} \psi_x^2 + \frac{R}{2(\gamma-1)} \eta_x^2 + \frac{\gamma_1}{2} z_x^2 \right) dx, \end{aligned} \tag{3.105}$$

and

$$\begin{aligned} \mathcal{D}_2 = & \frac{1}{4} \tilde{C}_2 \int_{\mathbb{R}} \left(\frac{\mu}{v} \psi_x^2 + \frac{\kappa}{v\vartheta} \eta_x^2 + \frac{\gamma_0}{8} \varphi(\theta) z^2 + \frac{\gamma_0}{4} \frac{d}{v} z_x^2 \right) dx + \tilde{C}_3 \int_{\mathbb{R}} \frac{\tilde{p}}{4\tilde{v}} \phi_x^2 dx \\ & + \int_{\mathbb{R}} \left(\frac{\mu}{4v} \psi_{xx}^2 + \frac{\kappa}{4v} \zeta_{xx}^2 + \frac{\varphi(\theta)\gamma_1}{4} z_x^2 + \frac{\lambda d \gamma_1}{4v^2} z_{xx}^2 \right) dx. \end{aligned} \tag{3.106}$$

Then, it follows from Lemmas 3.3-3.5 and the smallness of $\bar{\delta}$ and ε_0 that

$$\mathcal{E}_{2t} + \mathcal{D}_2 \leq C\bar{\delta}(1+t)^{-1} \mathcal{D}_1 + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \tag{3.107}$$

4 Decay Rates

In this section, we first show the L^2 -norm decay estimates of the solution $z(x, t)$. From (3.67), there exists a positive constant \tilde{C}_4 such that

$$\frac{d}{dt} \|z\|^2 + \tilde{C}_4 \|z\|^2 + \tilde{C}_4 \|z_x\|^2 \leq 0. \tag{4.1}$$

Multiplying (4.1) by $e^{\tilde{C}_4 t}$, it is obvious that

$$\frac{d}{dt} \left(e^{\tilde{C}_4 t} \|z\|^2 \right) + \tilde{C}_4 e^{\tilde{C}_4 t} \|z_x\|^2 \leq 0. \tag{4.2}$$

Integrating (4.2) with respect to t over $[0, t]$ yields

$$\|z\|^2 + \tilde{C}_4 \int_{\mathbb{R}} e^{-\tilde{C}_4(t-\tau)} \|z_x\|^2 dx \leq e^{-\tilde{C}_4 t} \|z_0\|^2. \tag{4.3}$$

By combining Lemmas 3.1-3.5, we get the time-decay rates of the solution to the nonlinear problem.

Proposition 4.1. *Under the assumptions of Theorem 1.1, it holds that*

$$\|(\phi, \psi, \zeta)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{4}} \quad \text{and} \quad \|z\|_{L^\infty} \leq C e^{-Ct}. \tag{4.4}$$

Proof. It follows from (3.38), (3.107) and (4.3) that

$$\begin{aligned} (\mathcal{E}_1 + \mathcal{E}_2)_t + \mathcal{D}_1 + \mathcal{D}_2 \leq & (C_0 \bar{\delta} + \frac{1}{4})(1+t)^{-1} (\mathcal{E}_1 + \mathcal{E}_2) \\ & + C_0 \bar{\delta} (1+t)^{-\frac{1}{2}} + C_0 (1+t) e^{-\tilde{C}_4 t} \|z_0\|^2, \end{aligned} \tag{4.5}$$

for some positive constant C_0 . Multiplying (4.5) by $(1+t)^{-C_0 \bar{\delta} - \frac{1}{4}}$ and using the Gronwall inequality, we get

$$\mathcal{E}_1 + \mathcal{E}_2 \leq C(\mathcal{E}_1(0) + \mathcal{E}_2(0) + \bar{\delta} + \|z_0\|^2)(1+t)^{\frac{1}{2}}, \tag{4.6}$$

and

$$\int_0^t (\mathcal{D}_1 + \mathcal{D}_2) dt \leq C(\mathcal{E}_1(0) + \mathcal{E}_2(0) + \bar{\delta} + \|z_0\|^2)(1+t)^{\frac{1}{2}}, \tag{4.7}$$

if $C_0 \bar{\delta} < \frac{1}{4}$. Since $\mathcal{E}_1 + \mathcal{E}_2 \geq C_4 \|(\Phi, \Psi, W)\|^2$ from some positive constant C_4 , it is obvious that (4.6) implies that

$$\|(\Phi, \Psi, W)\|^2 \leq C(\mathcal{E}_1(0) + \mathcal{E}_2(0) + \bar{\delta} + \|z_0\|^2)(1+t)^{\frac{1}{2}}. \tag{4.8}$$

On the other hand, multiplying (3.107) by $1+t$, we have

$$[(1+t)\mathcal{E}_2]_t \leq C\bar{\delta}\mathcal{D}_1 + \mathcal{E}_2 + C\bar{\delta}(1+t)^{-\frac{1}{2}} \leq \mathcal{D}_1 + \mathcal{D}_2 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{4.9}$$

Integrating (4.9) with respect to t over $[0, t]$ and using (4.7) yields

$$\mathcal{E}_2 \leq C(\mathcal{E}_1(0) + \mathcal{E}_2(0) + \bar{\delta} + \|z_0\|^2)(1+t)^{-\frac{1}{2}}, \tag{4.10}$$

where we have used the fact that

$$\begin{aligned} \mathcal{E}_2 &\leq C\|(\phi, \psi, \eta, z)\|_{H^1}^2 \\ &\leq C(\|\Phi, \Psi, W\|_x)^2 + \|z\|^2 + \|(\phi, \psi, \eta, z)_x\|^2 + C\bar{\delta}(1+t)^{-\frac{3}{2}} \\ &\leq C\mathcal{D}_1 + C\mathcal{D}_2 + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{4.11}$$

Noticing the definition of \mathcal{E}_2 , we have that there exists a positive constant C_5 such that

$$\begin{aligned} \mathcal{E}_2 &\geq C_5\|(\phi, \psi, \eta, z)\|_{H^1}^2 \\ &\geq C_5(\|\Phi, \Psi, W\|_x)^2 + \|z\|^2 + \|(\phi, \psi, \eta, z)_x\|^2 - C_5\bar{\delta}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{4.12}$$

By the Sobolev inequality, (4.3) and (4.12), we have

$$\begin{aligned} \|(\phi, \psi, \zeta)\|_{L^\infty} &\leq C\|(\phi, \psi, \eta, z)\|_{H^1}^{\frac{1}{2}}\|(\phi, \psi, \eta, z)_x\|_{H^1}^{\frac{1}{2}} \\ &\leq C\mathcal{E}_2^{\frac{1}{2}} \leq C(\epsilon^2 + \bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \|z\|_{L^\infty} &\leq C\|z\|_{H^1}^{\frac{1}{2}}\|z_x\|_{H^1}^{\frac{1}{2}} \\ &\leq C\|z_0\|_{H^1}^{\frac{1}{2}}e^{-\frac{C_4 t}{2}}\mathcal{E}_2^{\frac{1}{4}}(1+t)^{-\frac{1}{8}} \\ &\leq C(\epsilon^2 + \bar{\delta})^{\frac{1}{2}}e^{-\frac{C_4 t}{2}}. \end{aligned} \tag{4.14}$$

Thus, we get (4.4). □

Remark 4.1. *It follows from the Sobolev inequality, (4.8), (4.10) and (4.12) that*

$$\begin{aligned} \|(\Phi, \Psi, W)\|_{L^\infty} &\leq C\|(\Phi, \Psi, W)\|_{H^1}^{\frac{1}{2}}\|(\Phi_x, \Psi_x, W_x)\|_{H^1}^{\frac{1}{2}} \\ &\leq C(\mathcal{E}_1(0) + \mathcal{E}_2(0) + \bar{\delta} + \|z_0\|^2)^{\frac{1}{2}}, \end{aligned} \tag{4.15}$$

which and $W = \frac{\gamma-1}{R}(\bar{W} - \tilde{u}\Psi)$ imply

$$\|(\Phi, \Psi, \bar{W})\|_{L^\infty} \leq C(\mathcal{E}_1(0) + \mathcal{E}_2(0) + \bar{\delta} + \|z_0\|^2)^{\frac{1}{2}}. \tag{4.16}$$

By (4.10), (4.12) and (4.15), we have

$$\|(\Phi, \Psi, W)\|_{L^\infty} + \|(\phi, \psi, \zeta, z)\|_{H^1} \leq C(\epsilon^2 + \bar{\delta})^{\frac{1}{2}}, \tag{4.17}$$

which implies that the a priori assumption (3.10) is verified. Therefore the proof of Theorem 1.1 is completed.

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