

Conformal-symmetry Wormholes Supported by a Perfect Fluid

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Abstract This paper presents a new wormhole solution by assuming that a homogeneously distributed fluid with equation of state $p = \omega\rho$ can be adapted to an anisotropic spacetime such as a wormhole and that this spacetime admits a one-parameter group of conformal motions. The pressure p in the equation of state becomes the lateral pressure p_t instead of the radial pressure p_r , as assumed in previous studies. Given that $p_t = \omega\rho$, p_r is then determined from the Einstein field equations. A wormhole solution can be obtained only if $\omega < -1$ or $0 < \omega < 1$. Since the former case corresponds to phantom dark energy, which has been the subject of earlier studies, we concentrate mainly on the latter. This case implies that given the above conditions, dark matter can support traversable wormholes.

Keywords: Wormholes, conformal symmetry, perfect fluid.

1 Introduction

Wormholes are hypothetical handles or tunnels in spacetime that connect different regions of our Universe or completely different universes altogether. That wormholes could be actual physical structures was first proposed by Morris and Thorne [1]. These could be described by the static and spherically symmetric line element

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

using units in which $c = G = 1$. Here $\Phi = \Phi(r)$ is referred to as the *redshift function*, which must be everywhere finite to avoid an event horizon. The function $b = b(r)$ is called the *shape function* since it helps to determine the spatial shape of the wormhole when viewed, for example, in an embedding diagram [1]. The spherical surface $r = r_0$ is the radius of the *throat* of the wormhole. Here $b = b(r)$ must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$, and $b'(r_0) < 1$, usually called the *flare-out condition*. This condition can only be satisfied by violating the null energy condition (NEC), defined as follows: for the stress-energy tensor $T_{\alpha\beta}$, we must have

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0$$

for all null vectors k^α . For Morris-Thorne wormholes, matter that violates the NEC is called "exotic."

The equation of state (EoS) for a standard perfect fluid, $p = \omega\rho$, $0 < \omega < 1$, was studied a long time ago by Chandrasekhar [2]. The realization that the Universe is undergoing an accelerated expansion [3, 4] has led to the value of $\omega < -1/3$ due to the Friedmann equation $\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)$. This case is known as *quintessence dark energy*. The value of $\omega = -1$ corresponds to the existence of Einstein's cosmological constant [5]. The case that has attracted the most attention in wormhole physics is $\omega < -1$, referred to as *phantom dark energy* since this case leads to a violation of the NEC: given the null vector $(1, 1, 0, 0)$, $p + \rho = -\omega\rho + \rho < 0$. The EoS $p = \omega\rho$, $0 < \omega < 1$, refers to ordinary (baryonic) matter, as well as to dark matter.

In this paper we also make use of conformal symmetry, the existence of a conformal Killing vector ξ defined by the action of \mathcal{L}_ξ on the metric tensor

$$\mathcal{L}_\xi g_{\mu\nu} = \psi(r) g_{\mu\nu}; \quad (2)$$

here \mathcal{L}_ξ is the Lie derivative operator and $\psi(r)$ is the conformal factor.

It is shown in this paper that the assumption of conformal symmetry implies that a Morris-Thorne wormhole is necessarily anisotropic. It is then assumed that a perfect-fluid distribution with EoS $p = \omega\rho$ can be adapted to an inhomogeneous spacetime such as a wormhole by letting p be the lateral pressure p_t instead of p_r , as assumed in previous studies [6, 7]. It is subsequently shown that a wormhole solution can exist only if $\omega < -1$ or $0 < \omega < 1$.

2 Conformal Killing Vectors and Wormhole Construction

This section consists of a brief discussion of the assumption that our spacetime admits a one-parameter group of conformal motions. First we need to recall that these are motions along which the metric tensor of the spacetime remains invariant up to a scale factor, which is equivalent to stating that there exists a set of conformal Killing vectors such that

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\eta\nu} \xi^\eta{}_{;\mu} + g_{\mu\eta} \xi^\eta{}_{;\nu} = \psi(r) g_{\mu\nu}, \tag{3}$$

where the left-hand side is the Lie derivative of the metric tensor and $\psi(r)$ is the conformal factor. Eq. (3) shows that the vector ξ characterizes the conformal symmetry since the metric tensor $g_{\mu\nu}$ is conformally mapped into itself along ξ . It must be emphasized that the assumption of conformal symmetry has proved to be fruitful in numerous ways, not only leading to new solutions but to new geometric and kinematical insights [8–13]. Another fairly recent discovery is that the Kerr black hole is conformally symmetric [14].

Exact solutions of traversable wormholes admitting conformal motions are discussed in Ref. [15] by assuming a noncommutative-geometry background. Two earlier studies assumed a non-static conformal symmetry [16, 17].

It is shown in Ref. [18] that the line element

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}$$

is particularly convenient for discussing the consequences of the conformal-symmetry assumption. In particular,

$$e^\nu = Cr^2 \tag{5}$$

and

$$e^\lambda = \psi^{-2}. \tag{6}$$

Moreover, the Einstein field equations are

$$\frac{1}{r^2}(1 - \psi^2) - \frac{(\psi^2)'}{r} = 8\pi\rho, \tag{7}$$

$$\frac{1}{r^2}(3\psi^2 - 1) = 8\pi p_r, \tag{8}$$

and

$$\frac{\psi^2}{r^2} + \frac{(\psi^2)'}{r} = 8\pi p_t. \tag{9}$$

It is clear from Eq. (5) that the wormhole spacetime cannot be asymptotically flat. So the wormhole material must be cut off at some $r = a$ and joined to an exterior Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{10}$$

We see from line element (1) that $M = \frac{1}{2}b(a)$. So for $e^\nu = Ca^2$, we have $Ca^2 = 1 - 2M/a$ and the constant of integration becomes

$$C = \frac{1}{a^2} \left(1 - \frac{b(a)}{a}\right). \tag{11}$$

3 Wormhole Solution

In this section we take a closer look at the EoS $p = \omega\rho$ as it relates to wormholes. In a cosmological setting, we are dealing with a homogeneously distributed fluid. On the other hand, for a given wormhole, the pressure may or may not be isotropic, but a wormhole admitting conformal motion is definitely not: suppose, on the contrary, that $p_r = p_t$. Then from Eqs. (8) and (9),

$$\frac{1}{r^2}(3\psi^2 - 1) = \frac{\psi^2}{r^2} + \frac{(\psi^2)'}{r}. \quad (12)$$

After simplifying, we obtain the differential equation

$$(\psi^2)' - \frac{2}{r}\psi^2 = -\frac{1}{r},$$

which is linear in ψ^2 and readily solved to obtain

$$\psi^2 = \frac{1}{2} + cr^2.$$

We will see a bit later that to obtain a wormhole solution, the equation must satisfy the initial condition $\psi^2(r_0) = 0$, where $r = r_0$ is the throat of the wormhole. The result is

$$\psi^2(r) = \frac{1}{2} - \frac{1}{2r_0^2}r^2. \quad (13)$$

Now observe that for $r > r_0$, $\psi^2(r_0) < 0$, which is impossible since $\psi(r)$ is a real-valued function.

Returning now to the EoS $p = \omega\rho$ describing a homogeneous distribution, it is emphasized in Refs. [6, 7] that an extension to an inhomogeneous spherically symmetric spacetime is possible by making p the radial pressure p_r , so that the transverse pressure p_t can then be determined from the Einstein field equations.

In this paper we follow the same strategy, but instead of assuming that $p_r = \omega\rho$, it is more convenient to use the EoS

$$p_t = \omega\rho; \quad (14)$$

(p_r is then determined by the Einstein field equations.) Substituting in this equation,

$$\frac{1}{8\pi} \left(\frac{\psi^2}{r^2} + \frac{(\psi^2)'}{r} \right) = \frac{\omega}{8\pi} \left[\frac{1}{r^2}(1 - \psi^2) - \frac{(\psi^2)'}{r} \right]. \quad (15)$$

After simplifying, we obtain the differential equation

$$(\psi^2)' + \frac{1}{r}\psi^2 = \frac{\omega}{1 + \omega} \frac{1}{r}, \quad (16)$$

again linear in ψ^2 . The solution is

$$\psi^2(r) = \frac{\omega}{1 + \omega} + \frac{c}{r}. \quad (17)$$

Comparing Eqs. (1) and (4), observe that

$$b(r) = r(1 - e^{-\lambda(r)}) = r[1 - \psi^2(r)] \quad (18)$$

by Eq. (6). The condition $b(r_0) = r_0$ at the throat implies that $\psi^2(r_0) = 0$, as noted earlier. It follows that the shape function is

$$b(r) = r \left(1 - \frac{\omega}{1 + \omega} + \frac{1}{r} \frac{r_0\omega}{1 + \omega} \right). \quad (19)$$

To check the flare-out condition, we need to find $b'(r_0)$:

$$b'(r_0) = 1 - \frac{\omega}{1 + \omega} + \frac{1}{r} \frac{r_0\omega}{1 + \omega} + r \left(-\frac{r_0\omega}{r^2(1 + \omega)} \right) \Big|_{r=r_0} = 1 - \frac{\omega}{1 + \omega}. \quad (20)$$

Now observe that $b'(r_0) < 1$ only if

$$-\frac{\omega}{1+\omega} < 0, \quad \omega \neq -1.$$

We conclude that $\omega < -1$ or $\omega > 0$, or, more completely,

$$\omega < -1 \quad \text{or} \quad 0 < \omega < 1. \tag{21}$$

According to Ref. [1], the flare-out condition is equivalent to $p_r + \rho < 0$, i.e., the NEC is violated for the null vector $(1, 1, 0, 0)$. Recalling the EoS $p_t = \omega\rho$, Eq. (21) with $\omega < -1$ implies that the NEC is violated for any null vector of the form

$$(1, 0, a, 1 - a), \quad 0 \leq a \leq 1.$$

For $0 < \omega < 1$, the NEC is actually met. It is readily checked, however, that

$$p_r + \rho|_{r=r_0} = -\frac{1}{8\pi} \frac{1}{r_0^2} \frac{\omega}{1+\omega} < 0$$

whenever

$$\omega < -1 \quad \text{or} \quad 0 < \omega < 1,$$

as before.

It is interesting to note that

$$p_r(r_0) = -\frac{1}{8\pi r_0^2},$$

which is independent of ω and coincides with $p_r(r_0)$ in Ref. [1].

4 The Shadow Universe

Even though our starting point was the lateral pressure in the EoS $p_t = \omega\rho$, it soon became apparent that $p_r + \rho < 0$ whenever $\omega < -1$ or $0 < \omega < 1$. In a cosmological setting, we normally associate $\omega < -1$ with phantom dark energy and $0 < \omega < 1$ with dark matter or normal (baryonic) matter. It is well known that phantom energy can support traversable wormholes since the NEC is violated [6, 7]. So in the present situation, the case $0 < \omega < 1$ is by far the more interesting.

For the EoS $p_t = \omega\rho$, the conditions $\omega < -1$ and $0 < \omega < 1$ cannot be met simultaneously. Well outside the galactic halo, however, the case $\omega < -1$ would apply since on large scales, the Universe is undergoing an accelerated expansion. Inside the galactic halo, on the other hand, dark matter dominates since the galaxies themselves do not participate in the expansion, being bound gravitationally by predominantly dark matter. So in the halo region, conformal-symmetry wormholes can be supported by dark matter. These wormholes would be part of what is often called the shadow universe, normally invisible to us. Detection of wormholes may nevertheless be possible by means of gravitational lensing [19–21].

For completeness let us note that it is shown in Ref. [22] that the Navarro-Frenk-White model can be used to show that dark matter can support traversable wormholes provided that we confine ourselves to the outer regions of the galactic halo. This restriction does not apply to the present study.

5 Conclusion

This paper discusses a new wormhole solution by making the common and presumably reasonable assumptions that a homogeneously distributed cosmic fluid can be adapted to an anisotropic spacetime such as a wormhole and that this spacetime admits a one-parameter group of conformal motions. The pressure p in the perfect-fluid EoS $p = \omega\rho$ becomes the lateral pressure p_t instead of the radial pressure p_r , as assumed in previous studies. Since $p_t = \omega\rho$ in this paper, p_r is determined from the Einstein field equations.

After showing that the assumption of conformal symmetry implies that any Morris-Thorne wormhole must be anisotropic, the EoS $p_t = \omega\rho$ is used to determine the conformal factor $\psi(r)$ and hence the shape

function $b = b(r)$. Neither could exist unless $\omega < -1$ or $0 < \omega < 1$. In both cases, $p_r + \rho < 0$, so that the NEC is violated.

Well outside the galactic halo region the EoS $p = \omega\rho$, $\omega < -1$, is normally interpreted as phantom dark energy, which is known to support traversable wormholes. So the more interesting case is $0 < \omega < 1$, representing primarily dark matter in the galactic halo region. This shows that given the above conditions, dark matter can support traversable wormholes.

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